Adaptive Multiple Importance Sampling for Structural Risk Assessment

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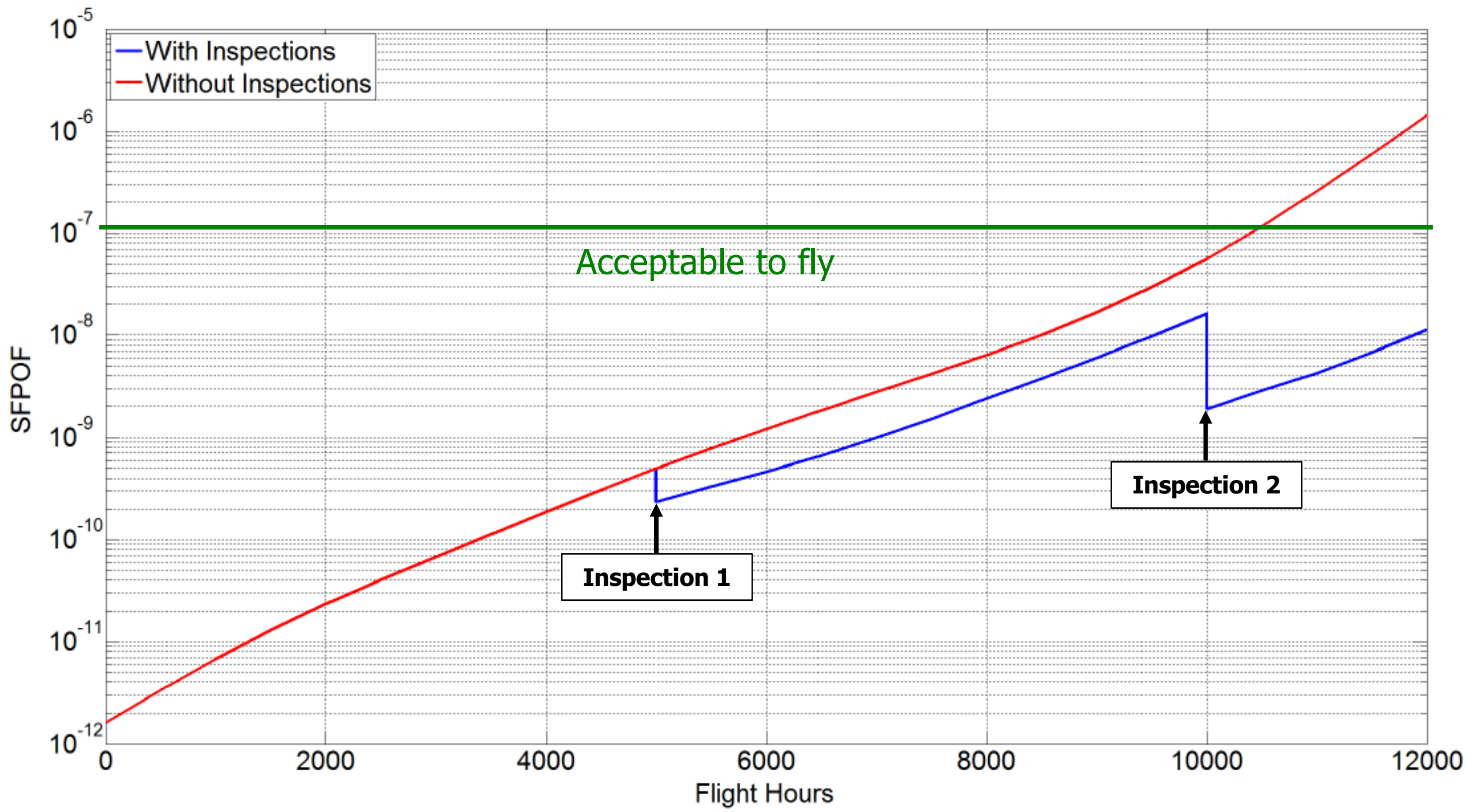
[1]N.Crosby, "Efficient Adaptive Importance Sampling Estimation of Time Dependent Probability of Failure with Inspections for Damage Tolerant Aircraft Structures," PhD dissertation, University of Texas at San Antonio, 2021

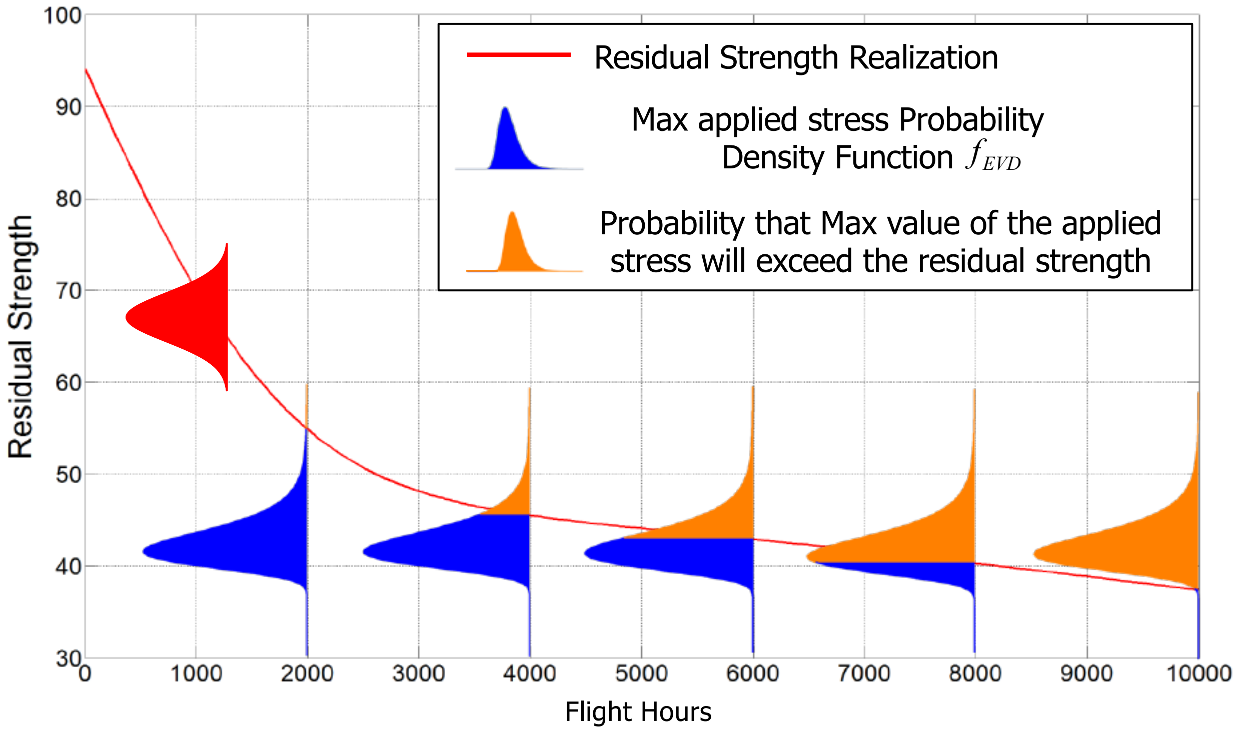
Probabilistic Risk Assessment

- Probabilistic Risk Assessment is an important tool for ensuring structural integrity of aircraft components.
- Based on the principles of probabilistic damage tolerance analysis.
- The Single Flight Probability-of-Failure is difficult to compute accurately and efficiently due to several challenges:
  1) Very small probabilities, e.g., 1E-7 or smaller
     - Standard Monte Carlo sampling is impractical
  2) Inspection and repair process results in multi-modal crack size distributions
     - FORM/SORM methods are impractical
  3) Inspection optimization requires multiple analyses
     - Efficient reanalyses are required

Probabilistic Risk Assessment



Probability of Failure Calculation



$$POF(t|a_i K_C) = 1 - F_{EVD}\left(\frac{K_C}{\beta(a(a_o,t))\sqrt{\pi a(a_o,t)}}\right)$$

Probability Equations

The probability-of-failure is the probability that maximum value of the applied stress (during the next flight) will exceed the residual strength σ_{RS} of the aircraft component.

$$POF_{Lincoln}(t) = P[\sigma_{Max} > \sigma_{RS}(t)] = \int_{\sigma_{Max} > \sigma_{RS}} [1 - F_{EVD}(\sigma_{RS}(\boldsymbol{x}, t_n))] f_X(\boldsymbol{x}) dx$$

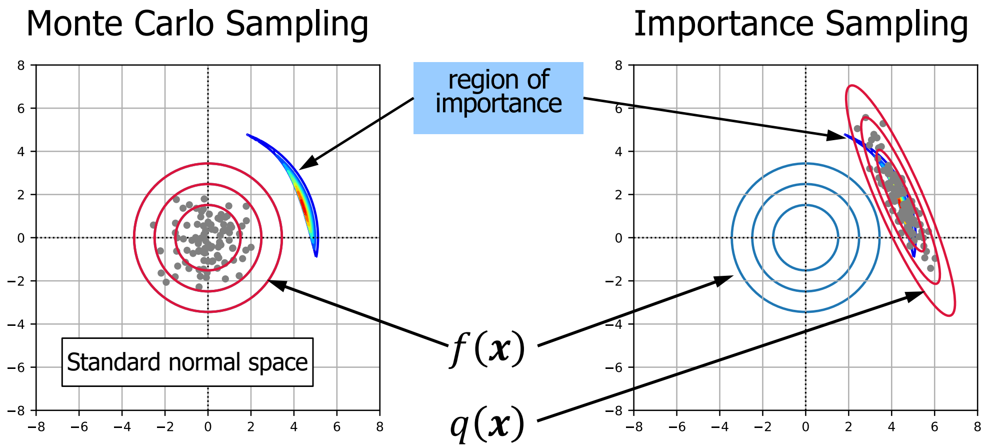
Other random variables

Survival to time t

$$POF_{Freudenthal}(t_n) = \frac{\int \left[\prod_{i=1}^{n-1} F_{EVD}\left(\sigma_{RS}(\boldsymbol{x}, t_i)\right)\right] [1 - F_{EVD}(\sigma_{RS}(\boldsymbol{x}, t_n)] f_X(\boldsymbol{x}) dx}{\int \prod_{i=1}^{n} F_{EVD}\left(\sigma_{RS}(\boldsymbol{x}, t_i)\right) f_X(\boldsymbol{x}) dx}$$

F_{EVD} – CDF of the maximum stress per flight (extreme value distribution)
σ_{RS} – residual strength

Importance Sampling



- Define $H(\boldsymbol{x};t) = 1 - F_{\mathrm{EVD}}\big(\sigma_{\mathrm{RS}}(\boldsymbol{x};t)\big)$

x – random variables:
- initial defect size
- fracture toughness
- dadN variability
- geometric parameters
- etc.

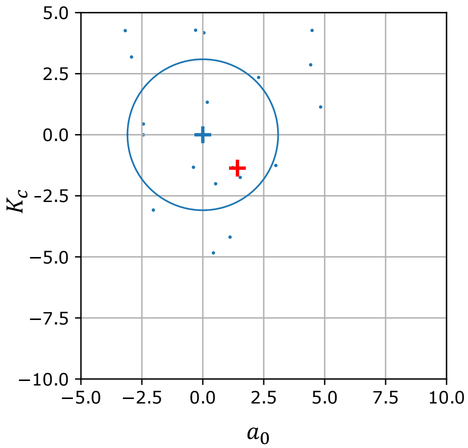
$$\mathrm{E}[H(\boldsymbol{x},t)] = \int H(\boldsymbol{x},t)\, f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \qquad \mathrm{E}[H(\boldsymbol{x},t)] = \int H(\boldsymbol{x},t) \frac{f(\boldsymbol{x})}{q(\boldsymbol{x})}\, q(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

$$\hat{E}[H(\boldsymbol{x},t)] = \frac{1}{N}\sum_i H(\boldsymbol{x}_i,t) \qquad \hat{E}[H(\boldsymbol{x},t)] = \frac{1}{N}\sum_i H(\boldsymbol{x}_i,t)\, w(\boldsymbol{x}_i)$$

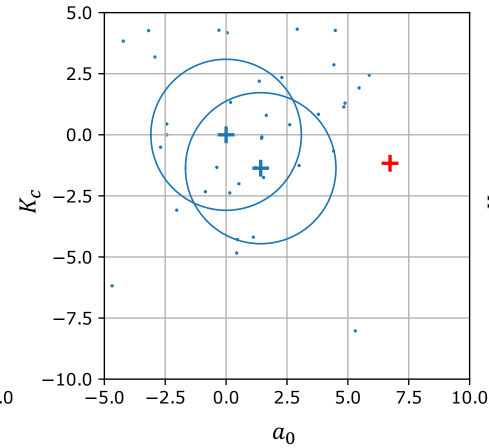
Draw samples from q

Importance weight
$w(\boldsymbol{x}_i) = f(\boldsymbol{x}_i) \,/\, q(\boldsymbol{x}_i)$

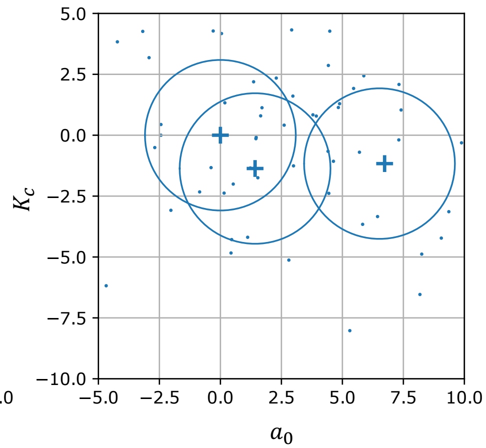
Phase I: Initialization



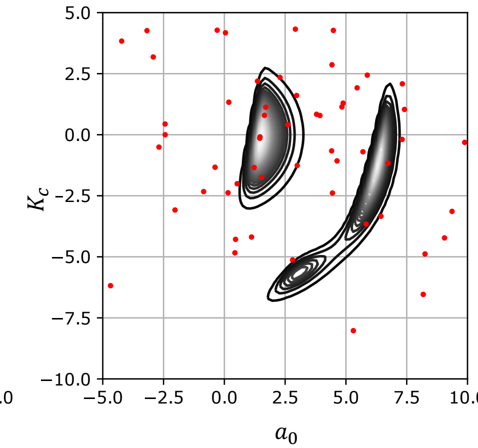
20 samples



40 samples



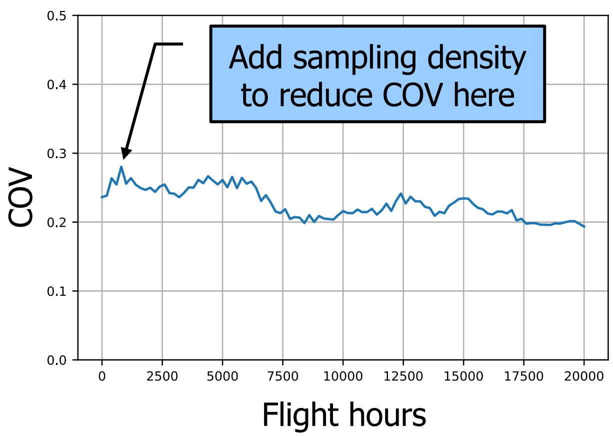
60 samples

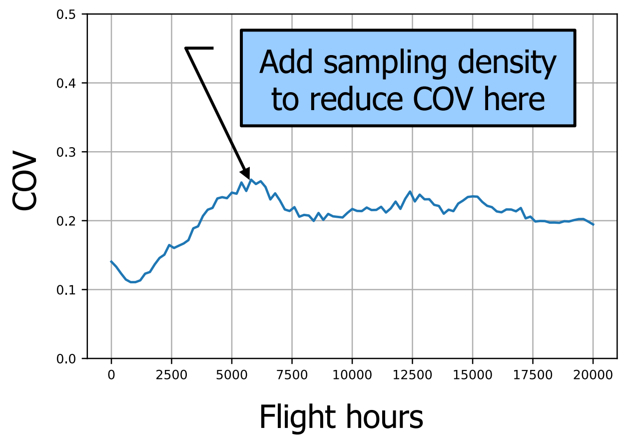


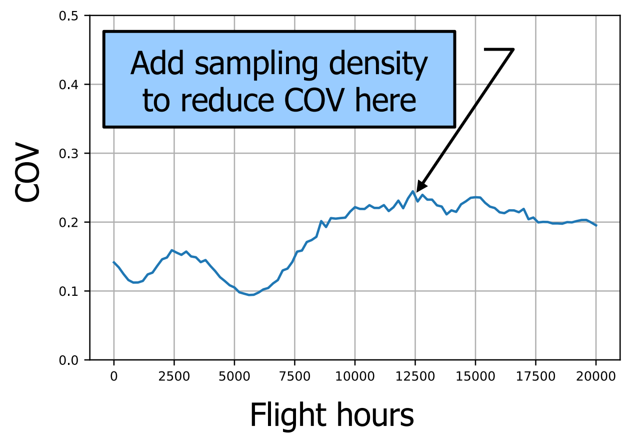
Comparison w important regions

- Goal is to locate important regions in standard normal space.
- Generate samples near and around the important regions for all evaluation times.
- Performs exploration to find the location of important regions.
- The adaptation phase will focus on determining the scale and shape of important regions.

Phase II: Adaptation



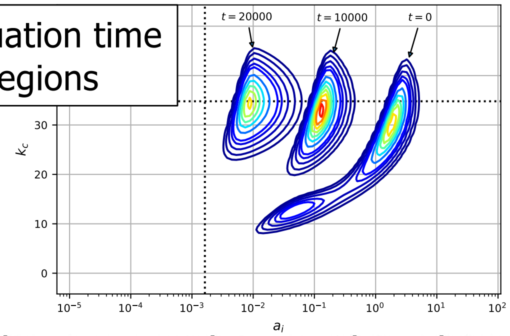




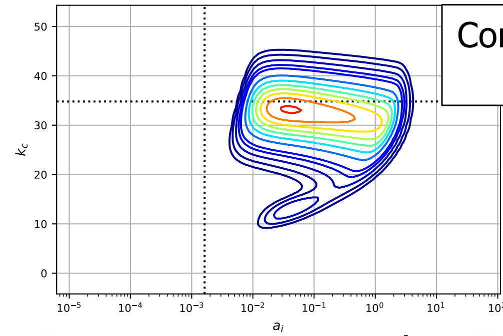
- Determine evaluation times at which to focus samples. Note, near-by times also obtain improved results.
- Use Coefficient of Variation (COV) which is a normalized error estimate.
- Ensures COV across all evaluation times is below a user-defined threshold.

Adaptive Multiple Importance Sampling Approach

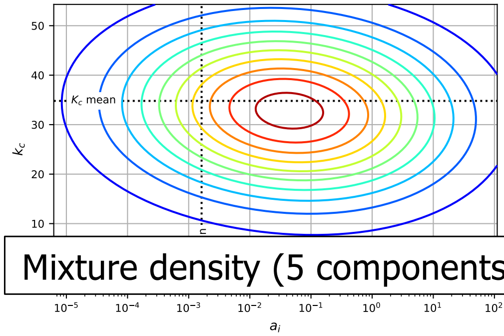
Individual evaluation time important regions



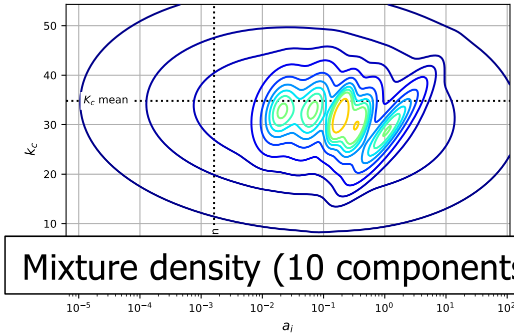
Combined important region



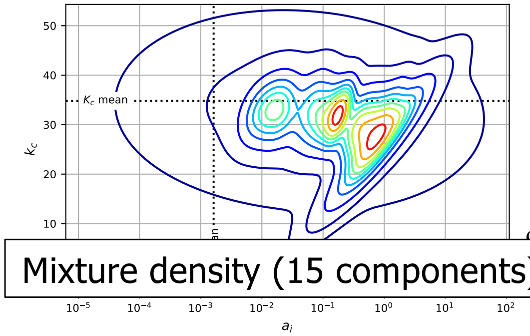
- Approximate the averaged or combined important region using a mixture density composed of multivariate normal sampling densities optimized for individual evaluation times.
- Key advantage is that samples can be used for more than one important region where regions overlap.



Mixture density (5 components)

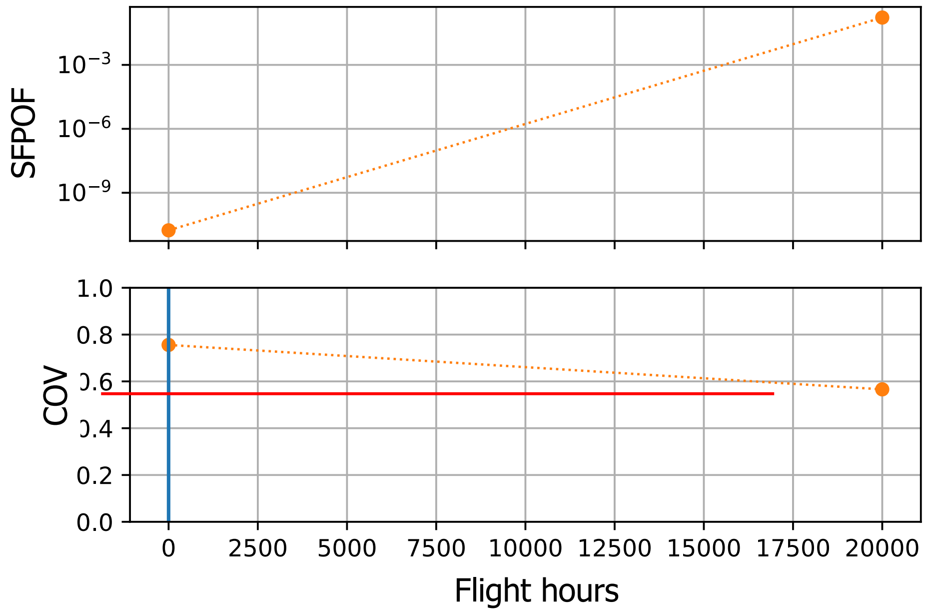


Mixture density (10 components)



Mixture density (15 components)

Academic Example



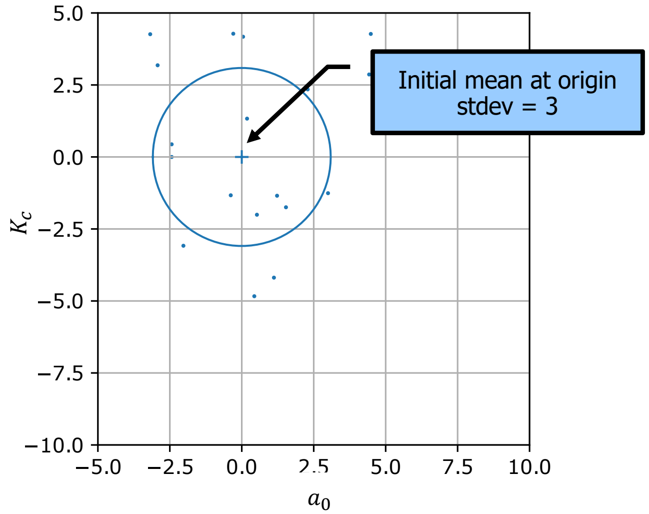
Random variables:

- ➢ Initial crack size
- ➢ Fracture toughness
- ➢ Max stress per flight

- 2 analysis times: t=0 and t=20000
- No inspections

Initialization 1/8

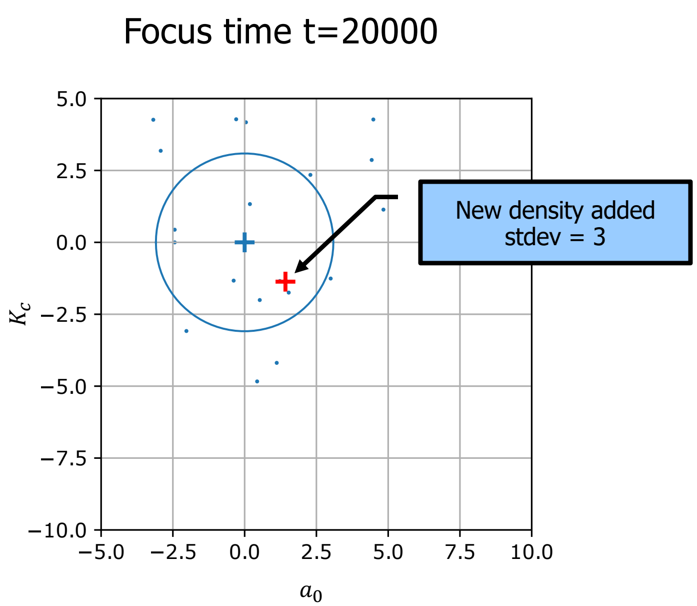
Initial focus @ Highest time, t=20000



- $N_{mix} = 1$
- Initial mixture set at the mean with covariance matrix $c_\sigma^2 I$.
- 20 samples generated.

- Initialize empty mixture density, starting $\mu^\star$ at origin, $k^\star$ points to last time in list of evaluation times.
- Add $(\mu^\star, c_\sigma^2\, I)$ to $\{\theta_{mix}\}$, set $\epsilon^\star = \epsilon_{KL}/10$ to check that $\mu^\star$ is not changing before moving to next evaluation time.
- Generate samples from $N(\mu^\star, c_\sigma^2\, I)$ and evaluate crack growth and response functions.

Initialization 2/8



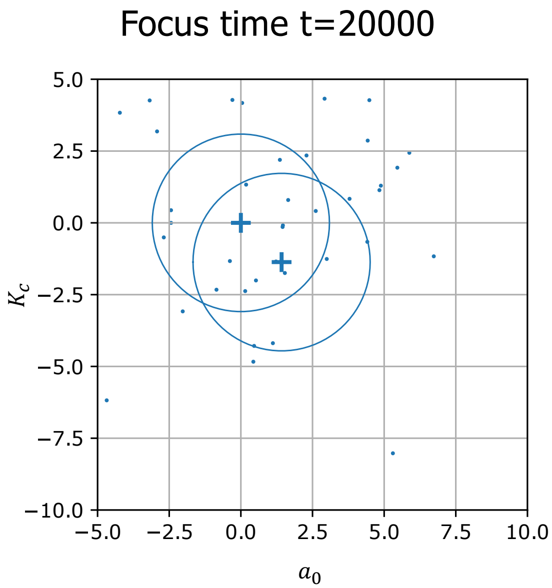
$N_{mix} = 1$ prior density

$$\epsilon^\star = 0.1 \text{ (convergence threshold)}$$
$$\mu^\star = (1.4, -1.4)$$
$$D_{min} = 0.22$$

- $D_{min} > \epsilon^*$ $\therefore$ new density added.
- $N_{mix} = 2$

- Using the 20 samples, calculate $\mu^\star$ using the cross-entropy method.
- Evaluate D_{min} from $\mu^\star$ to all component densities in the mixture.

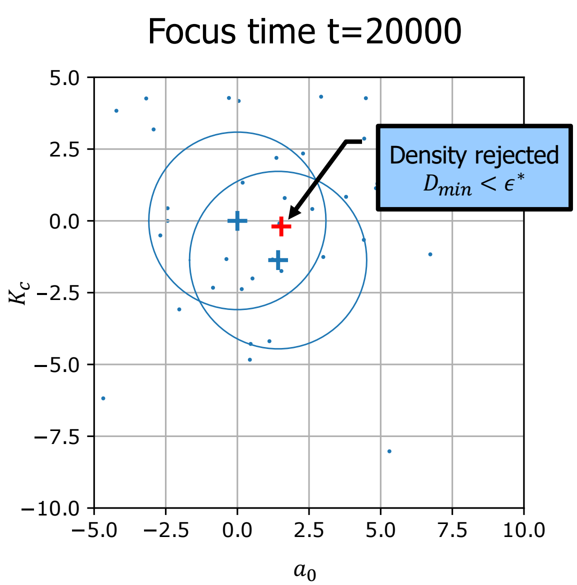
Initialization 3/8



- 20 new samples drawn from density #2.
- 40 samples total.

- Add $(\mu^{\star}, c_{\sigma}^{2}\, I)$ to $\{\theta_{mix}\}$, set $\epsilon^{\star} = \epsilon_{KL}/10$ to check that $\mu^{\star}$ is not changing before moving to next evaluation time.
- Generate samples from $N(\mu^{\star}, c_{\sigma}^{2}\, I)$ and evaluate crack growth and response functions.

Initialization 4/8



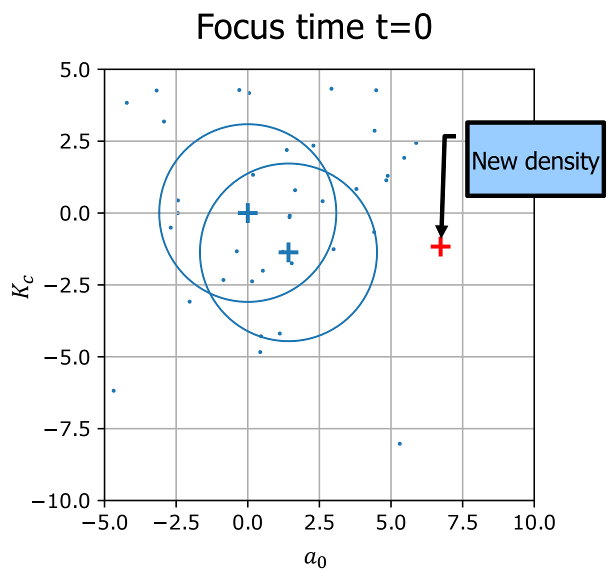
- $N_{mix} = 2$ prior densities.
- All 40 samples used to compute new location μ^*.

$$\epsilon^\star = 0.1$$
$$\mu^\star = (1.5, -0.2)$$
$$D_{min} = 0.08$$

- $D_{min} < \epsilon^*$
- Evaluation time satisfied. Moving to next time value.
- No samples generated since the new point (red) is close to another density ($D_{min} = 0.08$).

- Calculate $\mu^\star$ using standard weights.
- Evaluate D_{min} from $\mu^\star$ to all component densities in the mixture.

Initialization 5/8



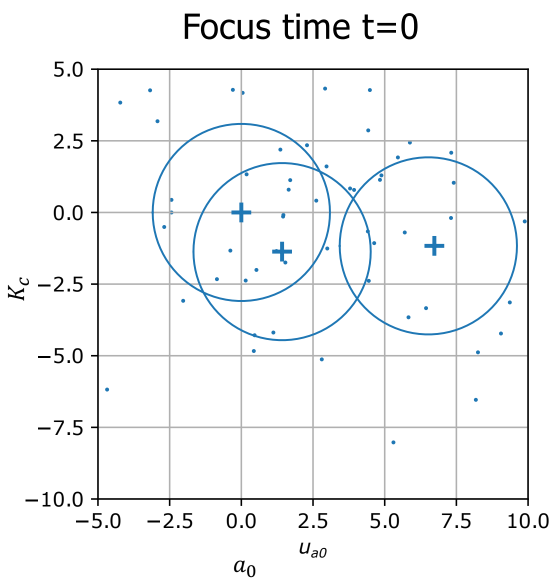
- $N_{mix} = 2$ prior densities.
- All 40 samples used to compute new location μ^*.

$$\epsilon^\star = 1.0$$
$$\mu^\star = (6.7, -1.2)$$
$$D_{min} = 1.6$$

- New density added centered at red cross.
- $N_{mix} = 3$

- The value H changes for the new time point.
- Calculate $\mu^\star$ using standard weights.
- Evaluate D_{min} from $\mu^\star$ to all component densities in the mixture.
- $D_{min} > \epsilon^\star$ so focus on the next evaluation time and locate its important region.

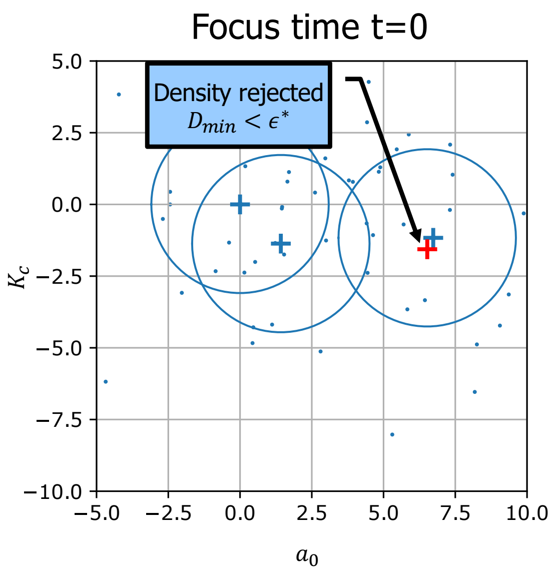
Initialization 6/8



- $N_{mix} = 3$ prior densities.
- Generate 20 new samples from density #3.
- Now 60 samples total.

- Add $(\mu^\star, c_\sigma^2\, I)$ to $\{\theta_{mix}\}$, set $\epsilon^\star = \epsilon_{KL}/10$ to check that $\mu^\star$ is not changing before moving to next evaluation time.
- Generate samples from $N(\mu^\star, c_\sigma^2\, I)$ and evaluate crack growth and response functions.

Initialization 7/8



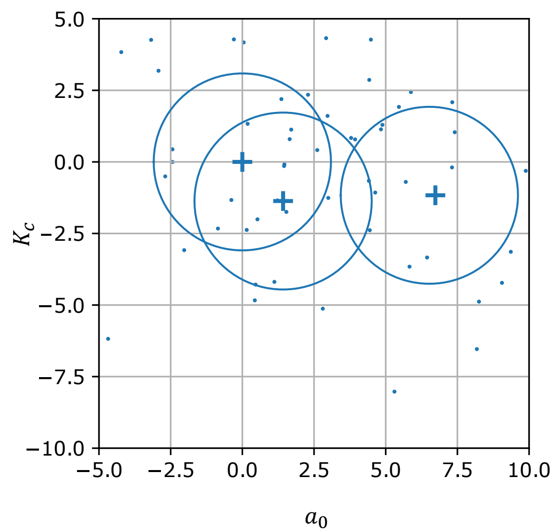
- $N_{mix} = 3$ prior densities.
- 60 samples used to compute new location μ^*.

$$\epsilon^\star = 0.1$$
$$\mu^\star = (6.5, -1.6)$$
$$D_{min} = 0.01$$

- $D_{min} < \epsilon^*$. Evaluation time satisfied. Moving to next time value.
- No samples generated since the new point (red) is close to another density ($D_{min} = 0.01$)

- Calculate $\mu^\star$ using standard weights.
- Evaluate D_{min} from $\mu^\star$ to all component densities in the mixture.
- $D_{min} < \epsilon^\star$ so focus on the next evaluation time and locate its important region

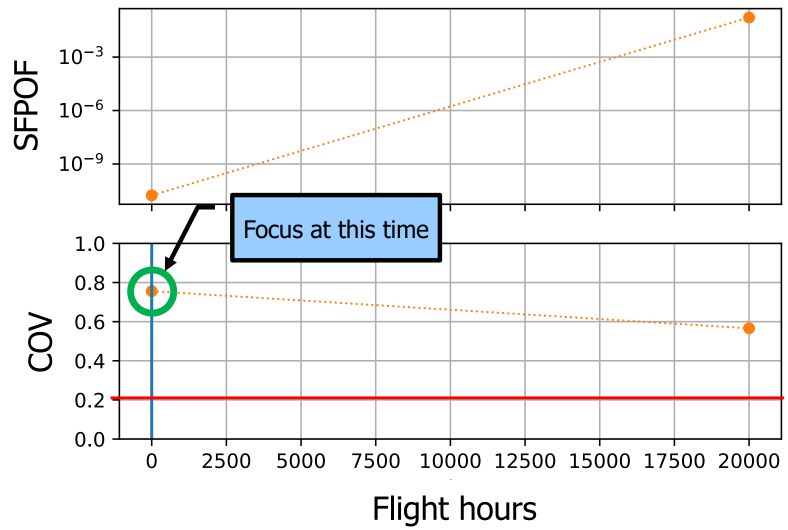
Initialization 8/8



- ➢ Initialization complete.
- ➢ 3 densities sufficient for initialization for 2 time points.
- ➢ Total of 60 samples.
- ➢ Crack growth and POF values saved for every sample.

- $k^\star = 0$, so initialization routine is finished.
- Return mixture density, realizations, crack growth evaluations and response function evaluations.

Adaptation Iteration 1/6



These equations are used to compute the POF and COV.

$$w_{bh}(x_{ij}) = \frac{q(x_{ij}, \theta_j)}{\sum_{l=1}^{N_m} (1 / N_m)\, q(x_{ij}, \theta_l)}$$

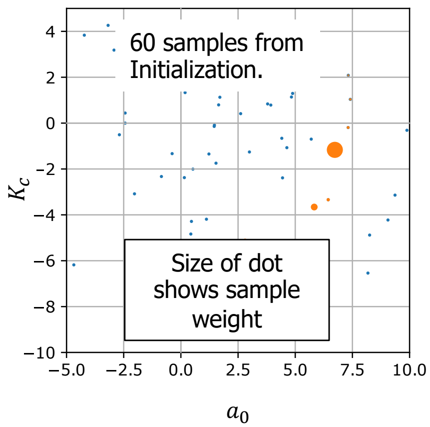
$$P_f(t_k) = \frac{1}{N} \sum_j \sum_i h_{ijk}\, w_{bh}(x_{ij})$$

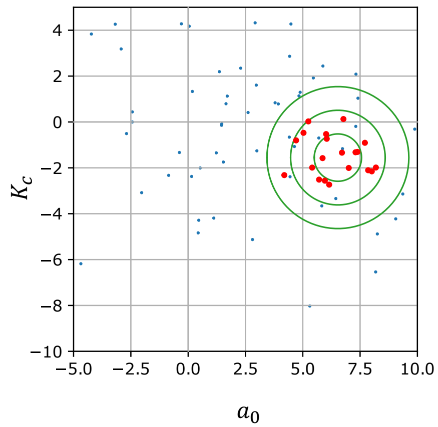
$$\mathrm{var}[P_f(t_k)] = \frac{1}{N} \sum_j \sum_i \left(h_{ijk}\, w_{bh}(x_{ij}) - P_f(t_k)\right)^2$$

$$\mathrm{cov}[P_f(t_k)] = \frac{\sqrt{\mathrm{var}[P_f(t_k)]/N}}{P_f(t_k)}$$

- Update balance heuristic importance weights
- Calculate estimates, estimator variances, and COVs
- Check exit condition (all COVs < ϵ_{cov}) or max iterations reached
- Select time with the highest COV

Adaptation Iteration 1/6





Focus time t=0

$$w_{std}(\boldsymbol{x}_{ij}) = h_{ijk}\frac{f(\boldsymbol{x}_{ij})}{q(\boldsymbol{x}_{ij};\boldsymbol{\theta}_j)}$$

$$n_{eff} = \frac{\left(\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}{\sum_j \sum_i \left(h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}$$

- 60 samples used to compute a new density.

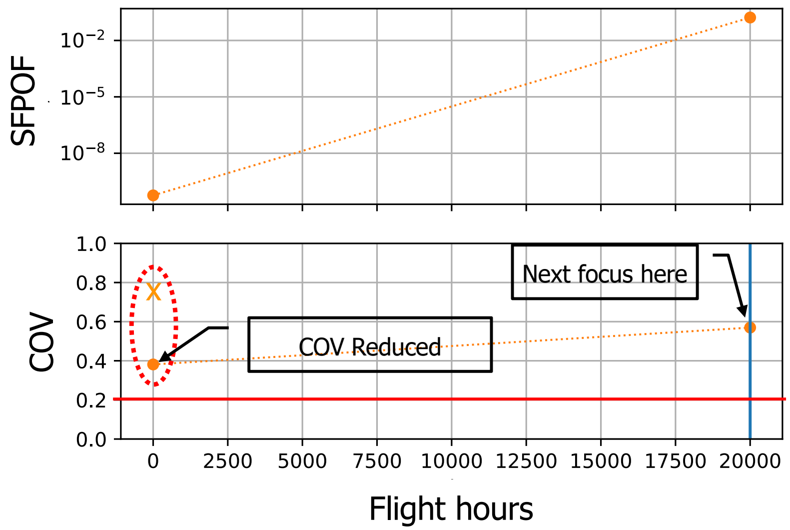
$$\mu^\star = \frac{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\boldsymbol{x}_{ij}}{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})}$$

$$\Sigma^\star = I$$

- New density added, $N_{mix} = 4$.
- 20 new samples generated (80 total).

- Calculate standard weights
- Effective sample size is 1, insufficient to update covariance matrix. $\Sigma^\star = I$.
- New density is $N(\mu^\star, I)$, generate samples and evaluate crack growth and response function

Adaptation Iteration 2/6



- POF and COV's computed from 80 samples.
- Weights updated using all 80 samples.
- All pofs and covs updated.
- COV reduced at t=0, moving to t=20,000.

$$w_{bh}(x_{ij}) = \frac{q(x_{ij}, \theta_j)}{\sum_{l=1}^{N_m} (1 / N_m)\, q(x_{ij}, \theta_l)}$$

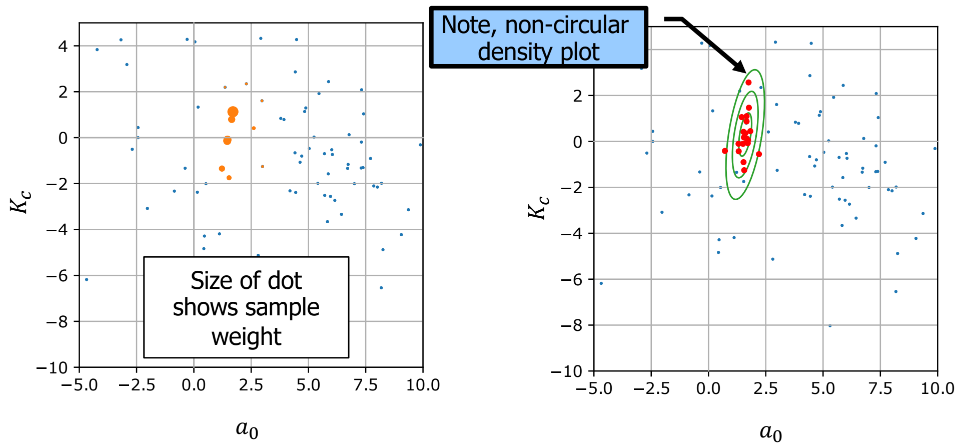
$$P_f(t_k) = \frac{1}{N} \sum_j \sum_i h_{ijk}\, w_{bh}(x_{ij})$$

$$\mathrm{var}[P_f(t_k)] = \frac{1}{N} \sum_j \sum_i \left(h_{ijk}\, w_{bh}(x_{ij}) - P_f(t_k)\right)^2$$

$$\mathrm{cov}[P_f(t_k)] = \frac{\sqrt{\mathrm{var}[P_f(t_k)]/N}}{P_f(t_k)}$$

- Update balance heuristic importance weights
- Calculate estimates, estimator variances, and COVs
- Check exit condition (all COVs < ϵ_{cov}) or max iterations reached
- Select time with the highest COV

Adaptation Iteration 2/6



Focus time t=20000

- New density added, $N_{mix} = 5$.
- 20 new samples generated.
- 100 samples total.

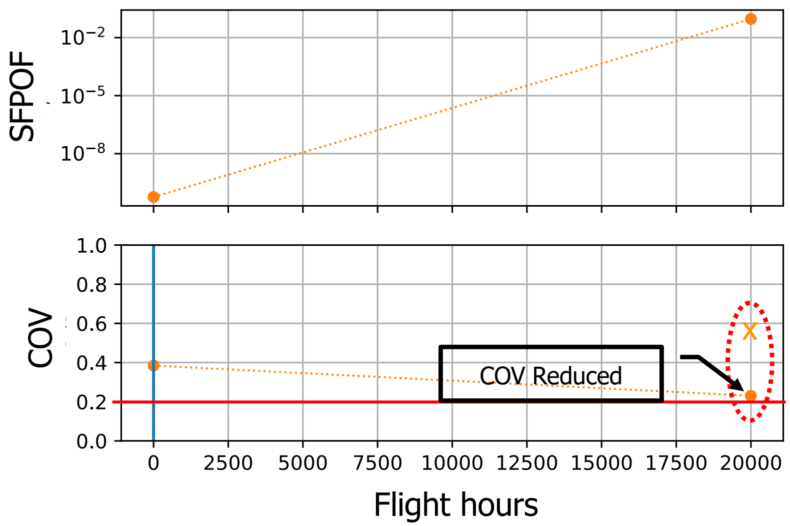
$$\mu^{\star} = \frac{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\boldsymbol{x}_{ij}}{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})}$$

$$\Sigma^{\star} = \frac{\sum_j \sum_i (\boldsymbol{x}_{ij} - \boldsymbol{\mu}_{\star})^{\mathsf{T}} h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})(\boldsymbol{x}_{ij} - \boldsymbol{\mu}_{\star})}{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})}$$

$$w_{std}(\boldsymbol{x}_{ij}) = h_{ijk} \frac{f(\boldsymbol{x}_{ij})}{q(\boldsymbol{x}_{ij}; \boldsymbol{\theta}_j)} \qquad n_{eff} = \frac{\left(\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}{\sum_j \sum_i \left(h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}$$

- Calculate standard weights.
- Effective sample size is 4, sufficient to update covariance matrix.
- New density is $N(\mu^{\star}, \Sigma^{\star})$, generate samples and evaluate crack growth and response function.

Adaptation Iteration 3/6



- 100 samples used.
- All pofs and covs updated.

$$w_{bh}(x_{ij}) = \frac{q(x_{ij}, \theta_j)}{\sum_{l=1}^{N_m} (1 / N_m)\, q(x_{ij}, \theta_l)}$$

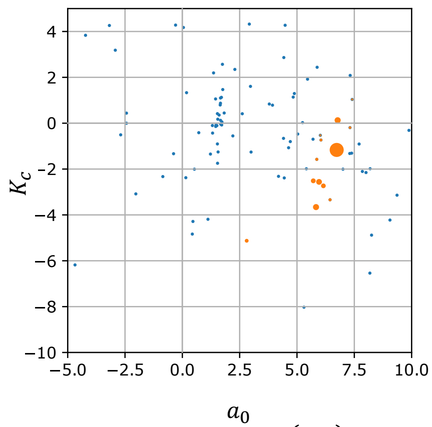
$$P_f(t_k) = \frac{1}{N} \sum_j \sum_i h_{ijk}\, w_{bh}(x_{ij})$$

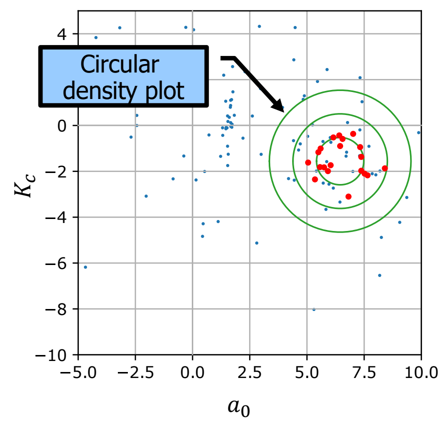
$$\mathrm{var}[P_f(t_k)] = \frac{1}{N} \sum_j \sum_i \left(h_{ijk}\, w_{bh}(x_{ij}) - P_f(t_k)\right)^2$$

$$\mathrm{cov}[P_f(t_k)] = \frac{\sqrt{\mathrm{var}[P_f(t_k)]/N}}{P_f(t_k)}$$

- Update balance heuristic importance weights.
- Calculate estimates, estimator variances, and COVs.
- Check exit condition (all COVs < ϵ_{cov}) or max iterations reached.
- Select time with the highest COV.

Adaptation Iteration 3/6





Focus time t=0

$$w_{std}(\boldsymbol{x}_{ij}) = h_{ijk} \frac{f(\boldsymbol{x}_{ij})}{q(\boldsymbol{x}_{ij}; \boldsymbol{\theta}_j)}$$

$$n_{eff} = \frac{\left(\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}{\sum_j \sum_i \left(h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}$$

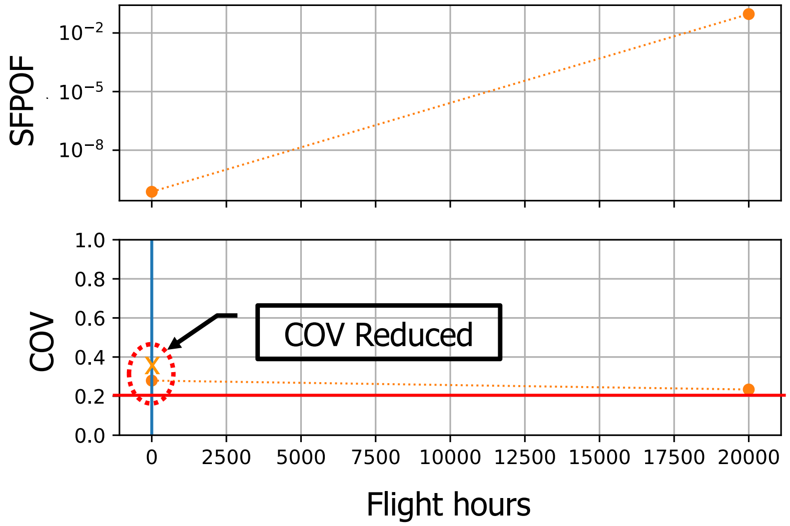
- New density added, $N_{mix} = 6$.
- 20 new samples generated.
- 120 samples total.

$$\mu^\star = \frac{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij}) \boldsymbol{x}_{ij}}{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})}$$

$$\Sigma^\star = I$$

- Calculate standard weights.
- Effective sample size is 2, insufficient to update covariance matrix.
- New density is $N(\mu^\star, I)$, generate samples and evaluate crack growth and response function.

Adaptation Iteration 4/6



- 120 samples used.
- All pofs and covs updated.

$$w_{bh}(x_{ij}) = \frac{q(x_{ij}, \theta_j)}{\sum_{l=1}^{N_m} (1 / N_m)\, q(x_{ij}, \theta_l)}$$

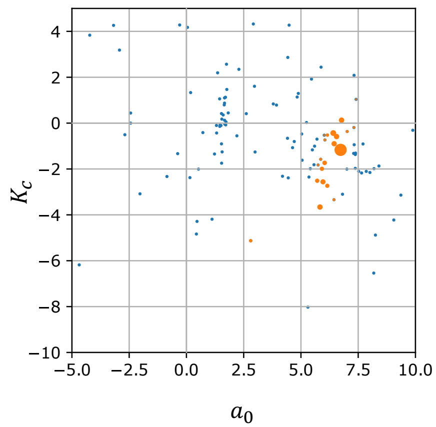
$$P_f(t_k) = \frac{1}{N} \sum_j \sum_i h_{ijk}\, w_{bh}(x_{ij})$$

$$\text{var}[P_f(t_k)] = \frac{1}{N} \sum_j \sum_i \left(h_{ijk}\, w_{bh}(x_{ij}) - P_f(t_k)\right)^2$$

$$\text{cov}[P_f(t_k)] = \frac{\sqrt{\text{var}[P_f(t_k)]/N}}{P_f(t_k)}$$

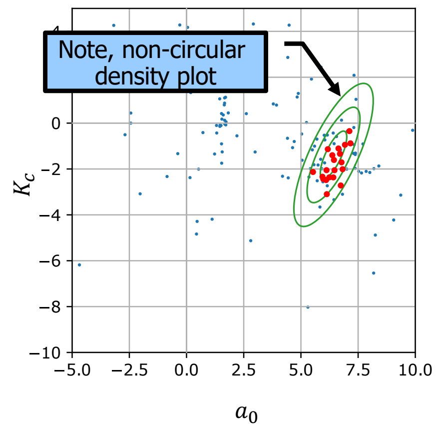
- Update balance heuristic importance weights
- Calculate estimates, estimator variances, and COVs
- Check exit condition (all COVs < ϵ_{cov}) or max iterations reached
- Select time with the highest COV

Adaptation Iteration 4/6



$$w_{std}(\boldsymbol{x}_{ij}) = h_{ijk} \frac{f(\boldsymbol{x}_{ij})}{q(\boldsymbol{x}_{ij}; \boldsymbol{\theta}_j)}$$

$$n_{eff} = \frac{\left(\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}{\sum_j \sum_i \left(h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}$$



Focus time t=0

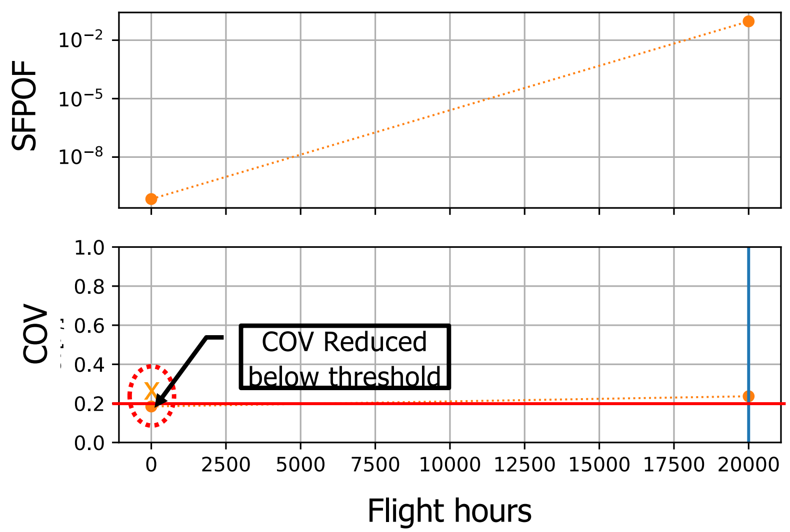
- New density added, $N_{mix} = 7$.
- 20 new samples generated.
- 140 samples total.

$$\mu^\star = \frac{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij}) \boldsymbol{x}_{ij}}{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})}$$

$$\Sigma^\star = \frac{\sum_j \sum_i (\boldsymbol{x}_{ij} - \boldsymbol{\mu}_\star)^\top h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})(\boldsymbol{x}_{ij} - \boldsymbol{\mu}_\star)}{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})}$$

- Calculate standard weights
- Effective sample size is 4, sufficient to update covariance matrix
- New density is $N(\mu^\star, \Sigma^\star)$, generate samples and evaluate crack growth and response function

Adaptation Iteration 5/6



- 140 samples used.
- All pofs and covs updated.
- COV(t=0) now below threshold.

$$w_{bh}(x_{ij}) = \frac{q(x_{ij}, \theta_j)}{\sum_{l=1}^{N_m} (1 / N_m)\, q(x_{ij}, \theta_l)}$$

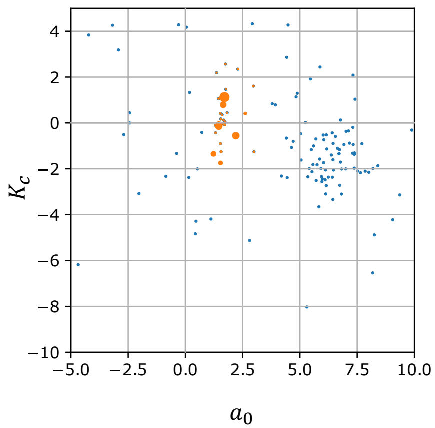
$$P_f(t_k) = \frac{1}{N} \sum_j \sum_i h_{ijk}\, w_{bh}(x_{ij})$$

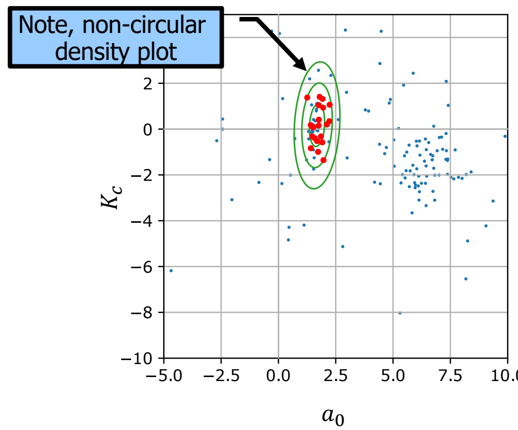
$$\mathrm{var}[P_f(t_k)] = \frac{1}{N} \sum_j \sum_i \left(h_{ijk}\, w_{bh}(x_{ij}) - P_f(t_k)\right)^2$$

$$\mathrm{cov}[P_f(t_k)] = \frac{\sqrt{\mathrm{var}[P_f(t_k)]/N}}{P_f(t_k)}$$

- Update balance heuristic importance weights.
- Calculate estimates, estimator variances, and COVs.
- Check exit condition (all COVs < ϵ_{cov}) or max iterations reached.
- Select time with the highest COV.

Adaptation Iteration 5/6





Focus time t=20000

- $N_{mix} = 8$.
- 20 new samples added.
- 160 samples total.
- All pofs and covs updated.

$$\mu^{\star} = \frac{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\boldsymbol{x}_{ij}}{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})}$$

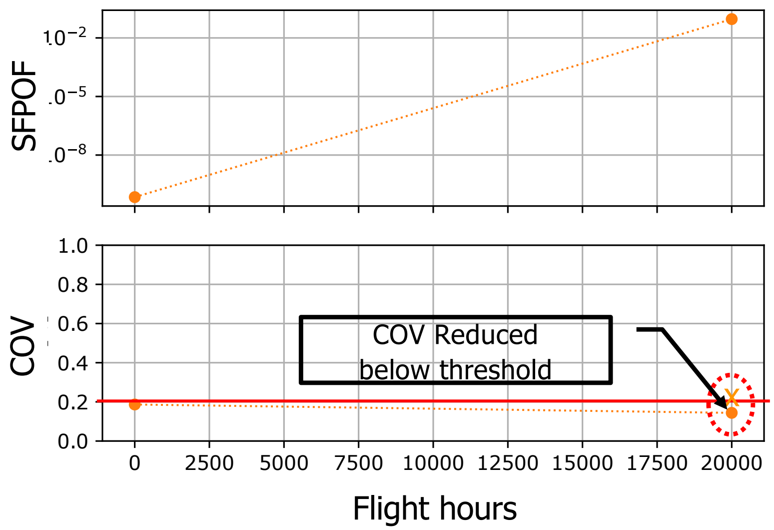
$$\Sigma^{\star} = \frac{\sum_j \sum_i (\boldsymbol{x}_{ij} - \boldsymbol{\mu}_{\star})^{\top} h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})(\boldsymbol{x}_{ij} - \boldsymbol{\mu}_{\star})}{\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})}$$

$$w_{std}(\boldsymbol{x}_{ij}) = h_{ijk} \frac{f(\boldsymbol{x}_{ij})}{q(\boldsymbol{x}_{ij}; \boldsymbol{\theta}_j)}$$

$$n_{eff} = \frac{\left(\sum_j \sum_i h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}{\sum_j \sum_i \left(h_{ijk}\, w_{std}(\boldsymbol{x}_{ij})\right)^2}$$

- Effective sample size is 6, sufficient to update covariance matrix.
- New density is $N(\mu^{\star}, \Sigma^{\star})$, generate samples and evaluate crack growth and response function.

Adaptation Iteration 6/6



- 160 samples used.
- All COVs below threshold.
- CG results saved for all 160 samples.

$$w_{bh}(x_{ij}) = \frac{q(x_{ij}, \theta_j)}{\sum_{l=1}^{N_m} (1 / N_m)\, q(x_{ij}, \theta_l)}$$

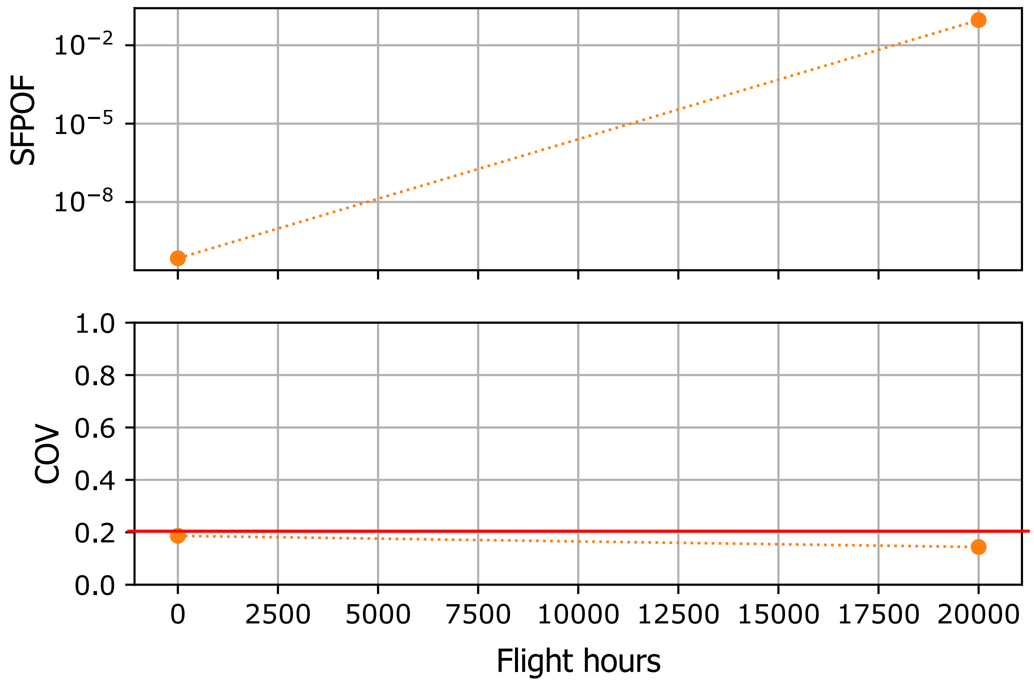
$$P_f(t_k) = \frac{1}{N} \sum_j \sum_i h_{ijk}\, w_{bh}(x_{ij})$$

$$\text{var}[P_f(t_k)] = \frac{1}{N} \sum_j \sum_i \left(h_{ijk}\, w_{bh}(x_{ij}) - P_f(t_k)\right)^2$$

$$\text{cov}[P_f(t_k)] = \frac{\sqrt{\text{var}[P_f(t_k)]/N}}{P_f(t_k)}$$

- Update balance heuristic importance weights.
- Calculate estimates, estimator variances, and COVs.
- Check exit condition (all COVs < ϵ_{cov}) or max iterations reached
- All COVs < ϵ_{cov}.

Academic Example Summary

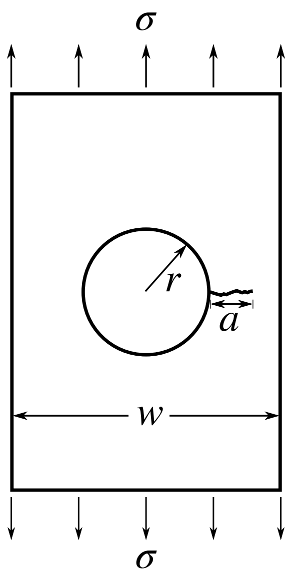


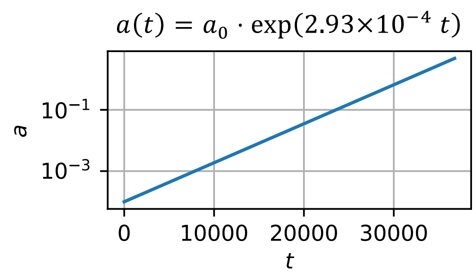
- SFPOF computed using 160 samples.
- (60 initialization, 100 adaptation).
- All COVs below user-defined threshold of 20%.

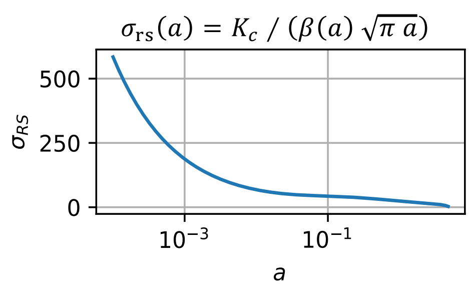
Example Problems

- Risk assessment handbook example using a closed-form crack growth equation.
- General aviation example with inspections.

Risk Assessment Handbook Problem





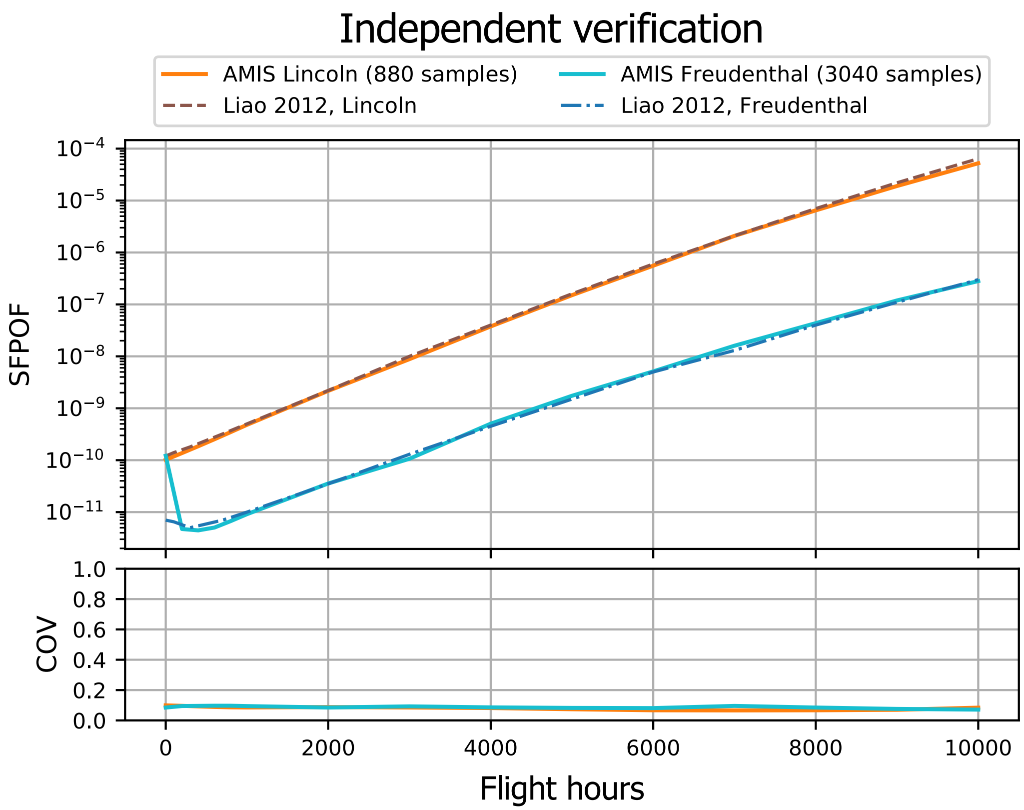


Parameter	Value
Width	Deterministic 10 in
Radius	Deterministic 0.125 in
Initial Crack Size	$LN(0.0032, 0.0047)$ in
Fracture Toughness	$N(34.8, 3.90)$ ksi $\sqrt{\text{in}}$
Maximum Stress per Flight	$W(5.0, 10.0, 5.0)$ ksi

$$\beta(a) = \underbrace{\left(0.6762 + \frac{0.8734}{0.3254 + a\ /\ R}\right)}_{\beta_{\text{hole}}} \cdot \underbrace{\sqrt{\sec\left(\frac{\pi\ (R + a)}{W}\right)}}_{\beta_{\text{width}}}$$

Tuegel et al., Aircraft structural reliability and risk analysis handbook volume 1: Basic analysis methods., Technical report, Air Force Research Lab, Wright-Patterson AFB, OH, Aerospace Systems Dir, 2013

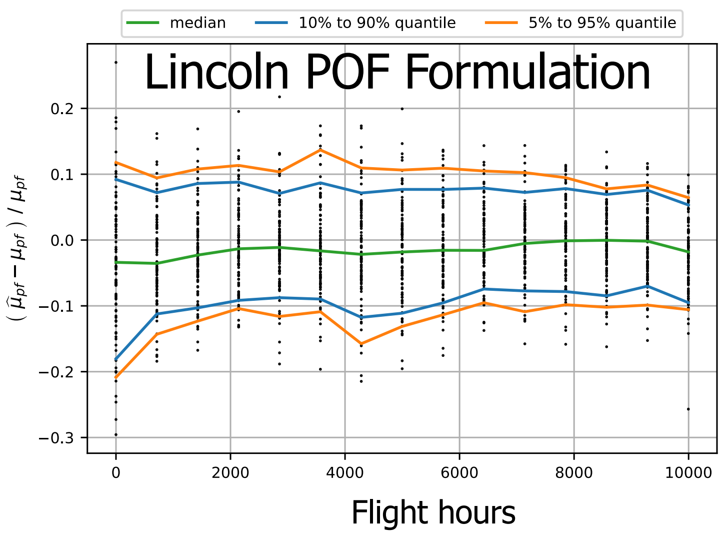
POF Results

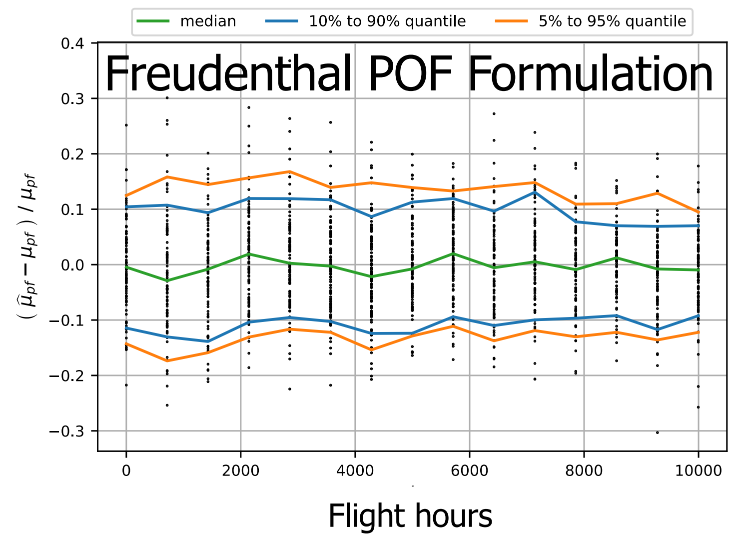


- 15 evaluation times
- COV threshold 0.1
- Lincoln Formulation
  - (assumes survival = 1 from flight 0 to flight t)
  - 80 samples per iteration
  - 11 iterations
  - 880 samples
- Freudenthal Formulation
  - (does not assume survival = 1 from flight 0 to flight t)
  - 160 samples per iteration
  - 19 iterations
  - 3040 samples

Liao M., Comparison of different single flight probability of failure (SFPOF) calculations for aircraft structural risk analysis. In Aircraft Airworthiness and Sustainment (AA&S) Conference, 2012

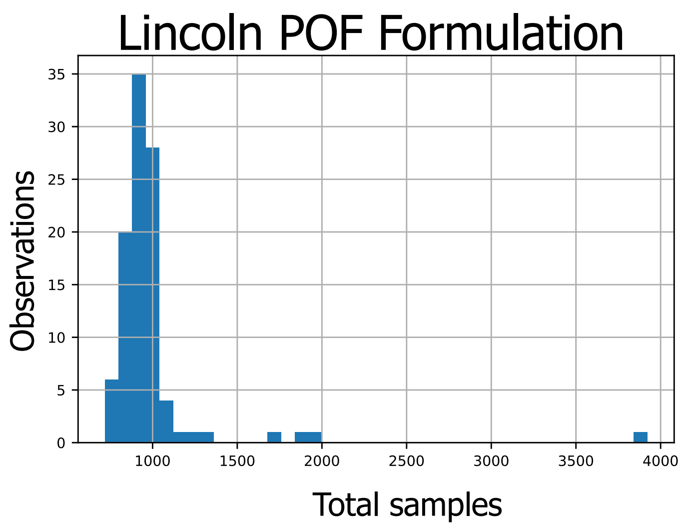
PDTA AMIS Accuracy of Error Estimates

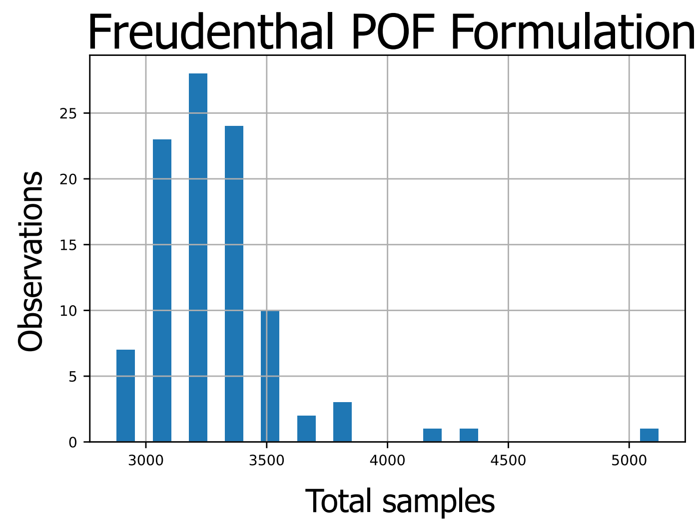




- Variations calculated for 100 PDTA AMIS runs.
- For both Lincoln and Freudenthal POF Formulations.
  - PDTA AMIS estimates are within the expected error bands, showing the sampling variance gives a good indication of estimator error.
  - PDTA AMIS median error is close to 0, showing the estimates are consistent.

PDTA AMIS Sampling Efficiency Variation





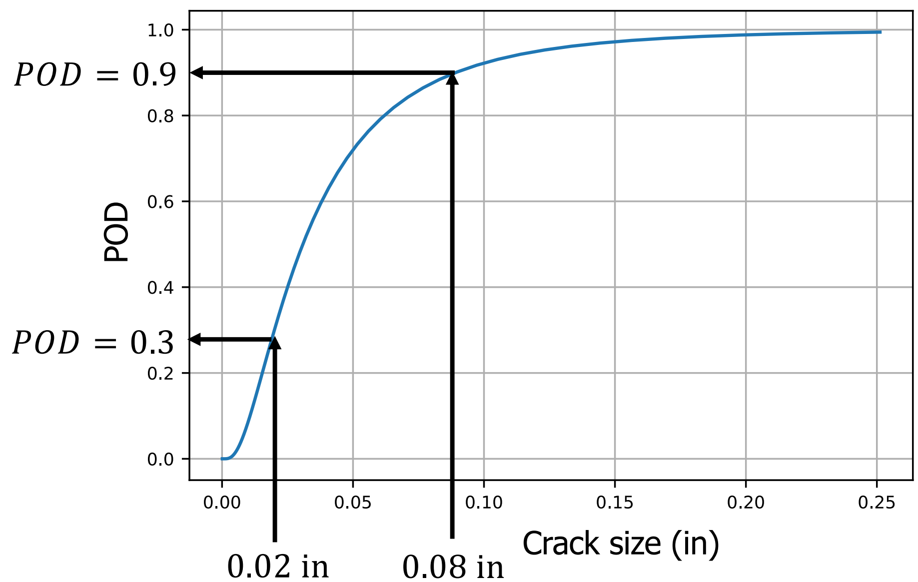
- Total samples collected from 100 PDTA AMIS runs
- Histograms show how many times PDTA AMIS finished the analysis using a given number of samples
- For both Lincoln and Freudenthal POF formulations
  - The right tail shows 5% of the runs can take several times longer than average

POF Inspections

Number of inspections before t

$$PND(t) = \prod_{i=1}^{N_I(t)} \left[1 - POD\big(a(\tau_i)\big)\right]$$

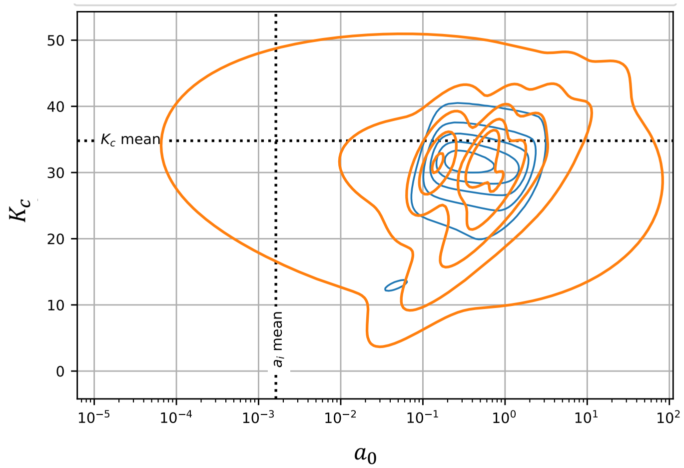
$$POF_{\text{no-surv}}(t) = \int PND\big(a(t)\big)\left[1 - F_{EVD}\big(\sigma_{RS}(t)\big)\right] f_{\mathbf{X}}\, \mathrm{d}\mathbf{x}$$



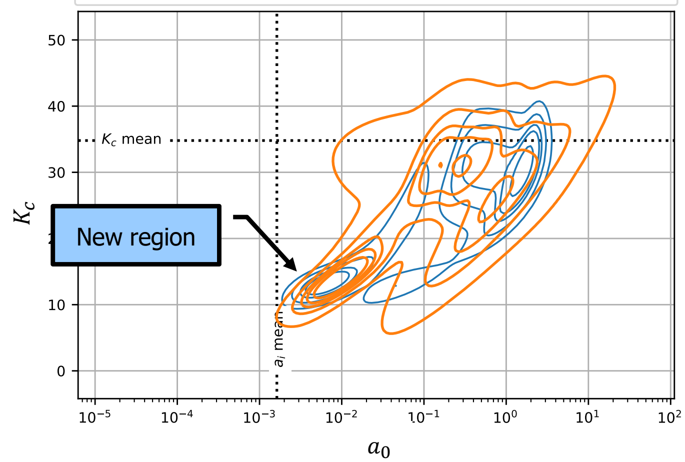
- Inspections are not deterministic – there is some probability of missing cracks
- In PDTA, this is modeled by reducing the probability of failure proportional to undetected cracks
- PND is the probability of not detecting a crack in any inspection(s) before t

Change in Combined Important Region Due to Inspection

Combined Important Region without inspection

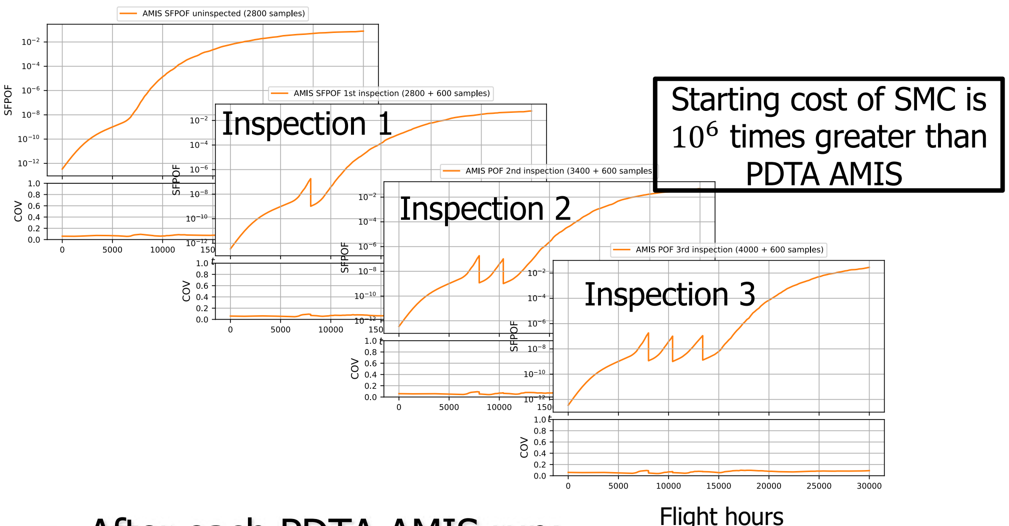


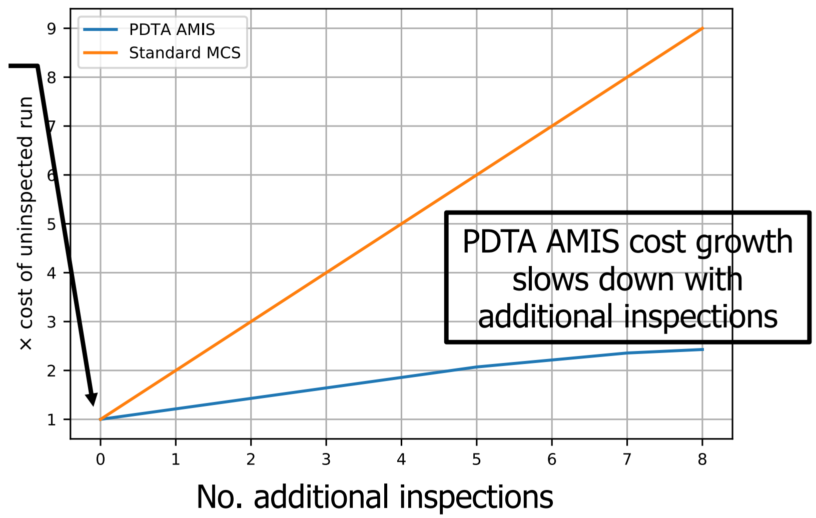
Combined Important Region with one Inspection



- Post-inspection, a new important region emerges around ($a_i = 0.007$, $k_c = 12.5$).
- Stored crack growth analyses reevaluated with the modified response function including an inspection provide a good general idea of the new important region location

Adding Inspections One-at-a-Time

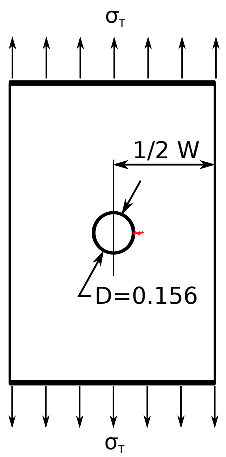


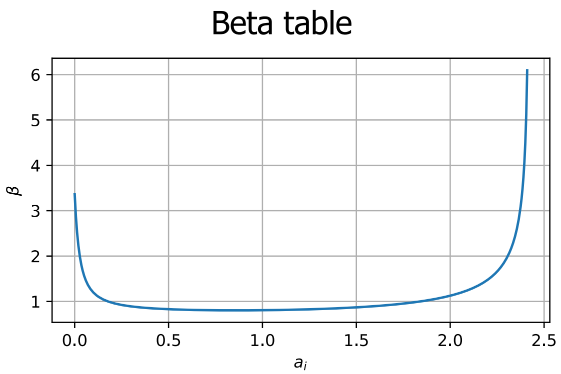


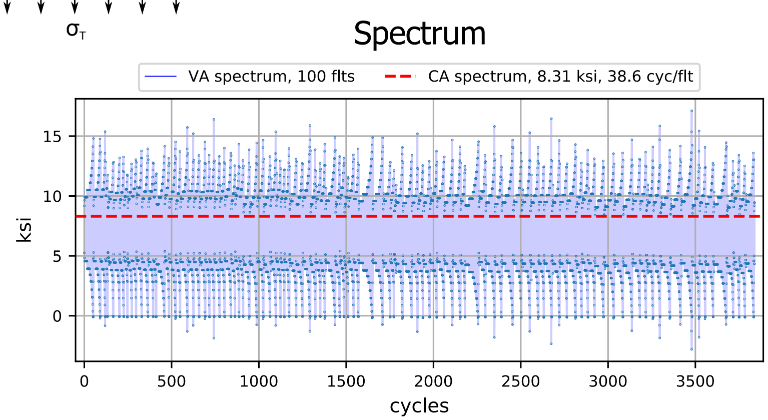
- After each PDTA AMIS run:
  - Update conditional POF, $H(\cdot)$, to include new inspection time in PND function
  - Recalculate $H(\cdot)$ for all samples over all times with existing crack growth evaluations
  - Re-run PDTA AMIS adaptation

- PDTA AMIS only has to add crack growth evaluations to adapt for the new inspection
- SMC must rerun all of the crack growth evaluations

General Aviation Example Problem

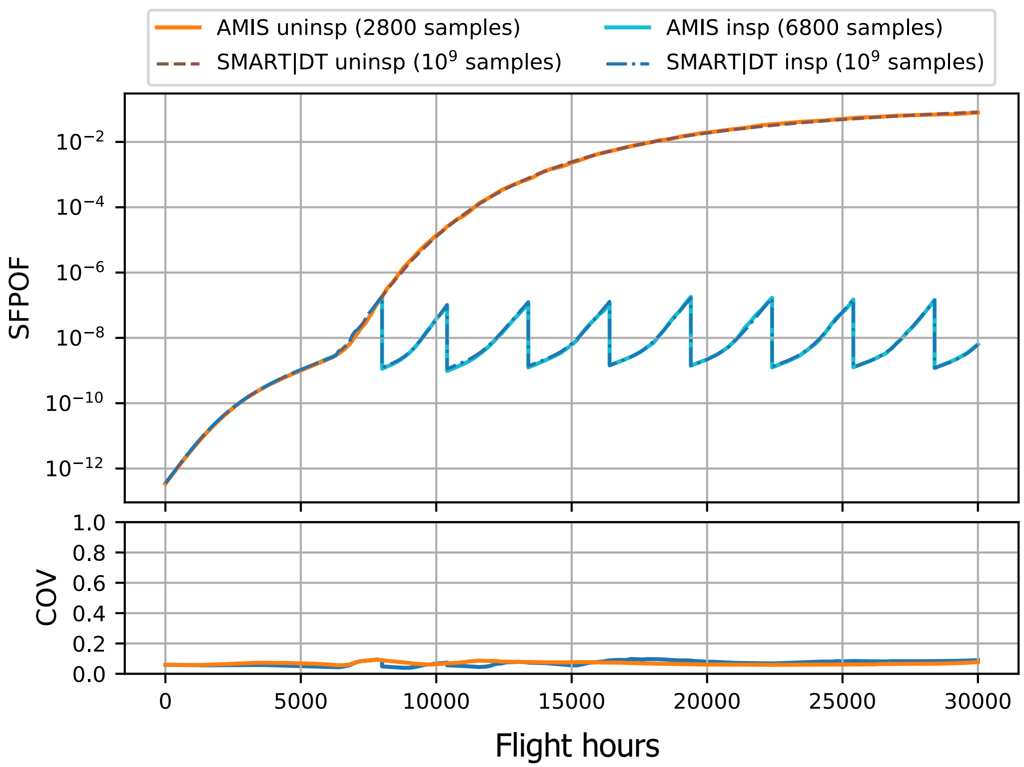






Parameter	Values
Width	Deterministic 5 in
Thickness	Deterministic 0.125 in
Log Paris Constant	$N(-9.0, 0.08)$
Paris Exponent	Deterministic 3.8
Initial Crack Size	$W(0.45, 4.17\times10^{-5})$ in
Fracture Toughness	$N(35.0, 3.5)$ ksi $\sqrt{\text{in}}$
Maximum Stress per Flight	$EVD(13.4, 1.3, 0.07)$ ksi
Probability of Detection	$LN(0.05, 0.065)$ in
Repair Quality (Crack Size)	Perfect

POF Results After Adding 8 Inspections



- PDTA AMIS
  - 2800 samples for uninspected POF
  - 6800 samples for inspected POF after adding 8 inspections
- PDTA AMIS in excellent agreement with SMC using 10^9 samples

Summary

- The AMIS algorithm estimates POF for risk assessment using 6 orders of magnitude fewer samples compared to standard Monte Carlo sampling for probabilities of 10^{-7} with COV of 0.1.
- Additionally, the PDTA AMIS algorithm enables storing and reusing crack growth analyses useful for the evaluation of multiple inspection schedules.

Future Developments

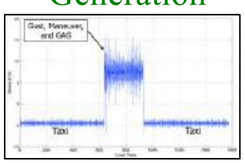
- Optimized inspection schedule
  - Determine the inspection times and inspection methods to keep the risk below a user-defined threshold with minimum cost.
- Probabilistic damage tolerance analysis of more realistic structures
  - Continuing damage, multisite damage, residual stresses, out-of-plane crack growth, etc.
- Approaches
  - Nasgro interface
  - Surrogate models
    - Machine learning approaches, e.g., Bingo software

Smart|DT Software

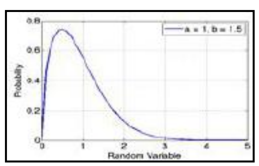
- Probabilistic risk assessment development has been funded by the US Federal Aviation Administration to develop the Smart|DT software.
- Available to the general public.
- Training presented annually and available online:
  - Aircraft Airworthiness Conference
  - https://smartdtsoftware.wixsite.com/smart

Web site link

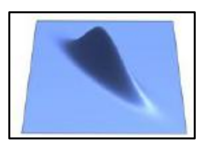
Loading Generation



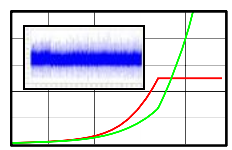
EVD Dist



dadN variability



HyperGrow



Interoperability

Scriptable

Acknowledgements

- Probabilistic Damage Tolerance-Based Maintenance Planning for Small Airplanes, Federal Aviation Administration, Grant 09-G-016
- Probabilistic Fatigue Management Program for General Aviation, Federal Aviation Administration, Grant 12-G-012
- Probabilistic Modeling of Random Variables and K-Solution Developments for General Aviation Grant 16-G-005
- Advances in Probabilistic Damage Tolerance Analysis using the Smart/DT Software, Cooperative Agreement 692M152140011
  - Sohrob Mattaghi (FAA Tech Center) – Program Manager
  - Michael Reyer (Kansas City Office) – Sponsor
  - Michael Gorelik – Chief Scientific and Technical Advisor for Fatigue and Damage Tolerance)