# Probabilistic Damage Tolerance Fundamentals


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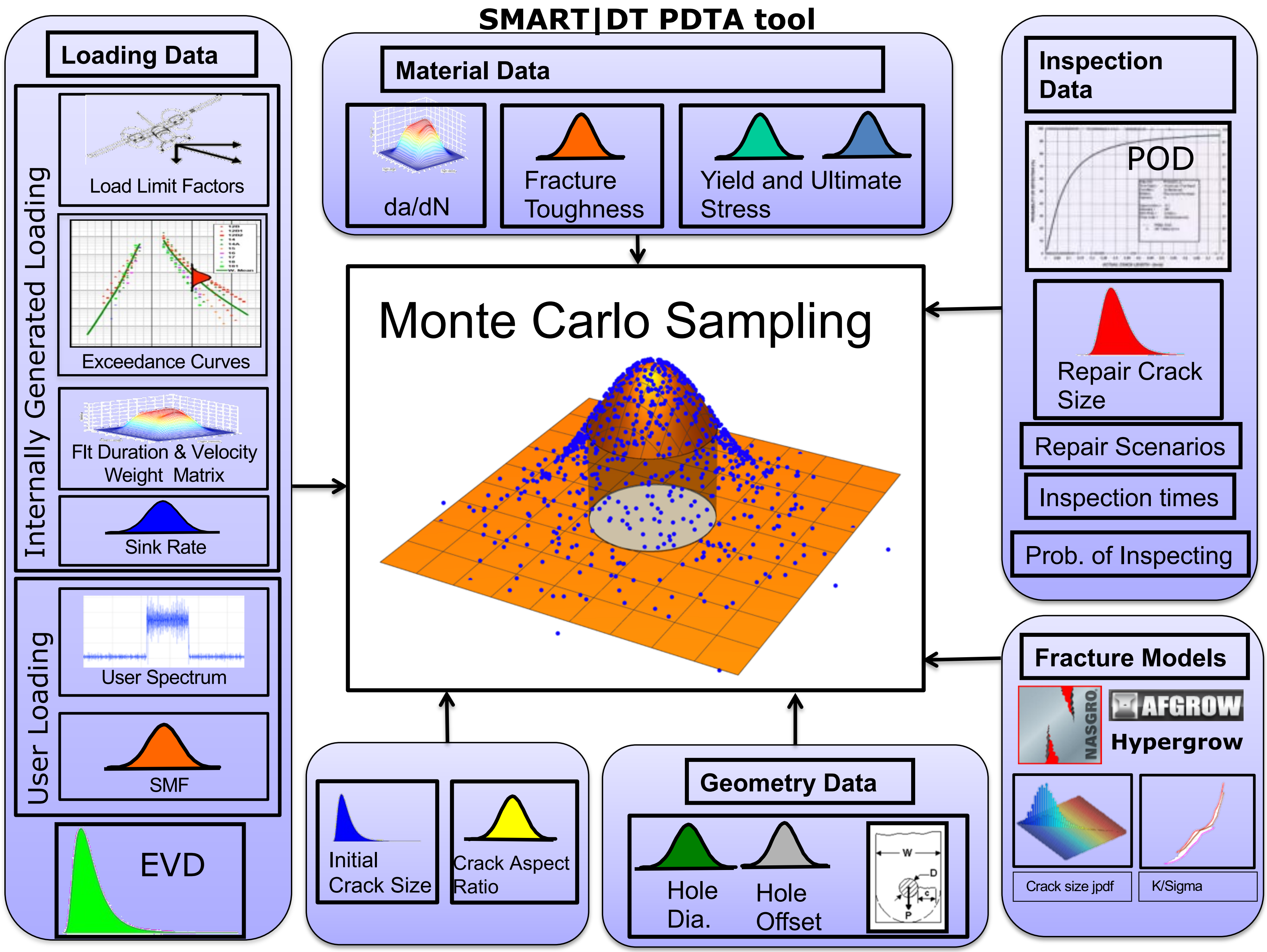
# PDTA Basics

## (Probabilistic Damage Tolerance Analysis)

- PDTA considers variation in:
  - Initial flaw size
  - Material properties
  - Geometry
  - Usage and loads
  - Inspection reliability and probability of detection
  - Repair
- PDTA results in:
  - Single flight probability of failure at any time during operation


SMART|DT PDTA tool
Loading Data
Internally Generated Loading
Load Limit Factors
Exceedance Curves
Flt Duration & Velocity Weight Matrix
Sink Rate
User Loading
User Spectrum
SMF
EVD
Material Data
da/dN
Fracture Toughness
Yield and Ultimate Stress
Monte Carlo Sampling
Inspection Data
POD
Repair Crack Size
Repair Scenarios
Inspection times
Prob. of Inspecting
Fracture Models
NASGRO
AFGROW
Hypergrow
Crack size jpdf
K/Sigma
Initial Crack Size
Crack Aspect Ratio
Geometry Data
Hole Dia.
Hole Offset

# Probability Equations

The probability-of-failure is the probability that maximum value of the applied stress (during the next flight) will exceed the residual strength $\sigma_{RS}$ of the aircraft component

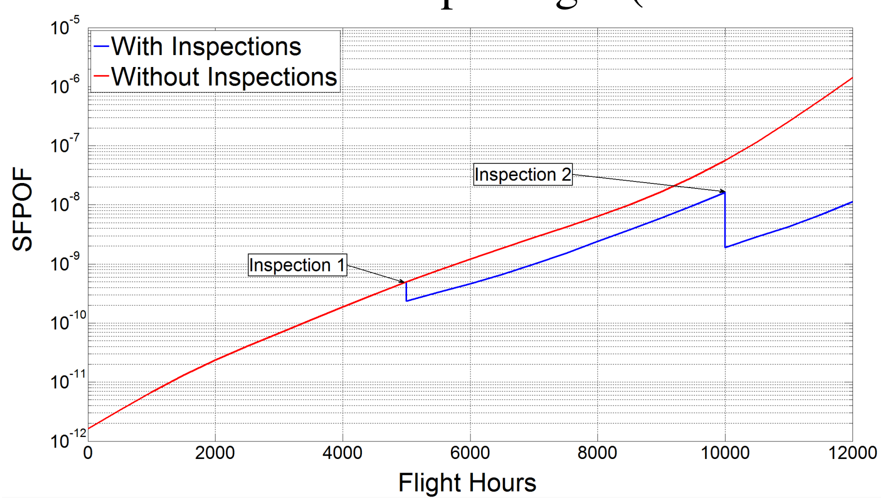
$$POF_{\text{no-surv}}(t) = P\left[\sigma_{Max} > \sigma_{RS}(t)\right] = \int \left[1 - F_{EVD}\left(\sigma_{RS}(t)\right)\right] f_{\mathbf{x}}(\mathbf{x})\, d\mathbf{x}$$

$$CTPOF(t) = \int \left[1 - \prod_{i=1}^{t} F_{EVD}\left(\sigma_{RS}(t_i)\right)\right] f_{\mathbf{x}}(\mathbf{x})\, d\mathbf{x}$$

$$POF_{\text{surv}}(t) = \int \left[\prod_{i=1}^{t-1} F_{EVD}\left(\sigma_{RS}(t_i)\right)\right] \left[1 - F_{EVD}\left(\sigma_{RS}(t)\right)\right] f_{\mathbf{x}}(\mathbf{x})\, d\mathbf{x}$$

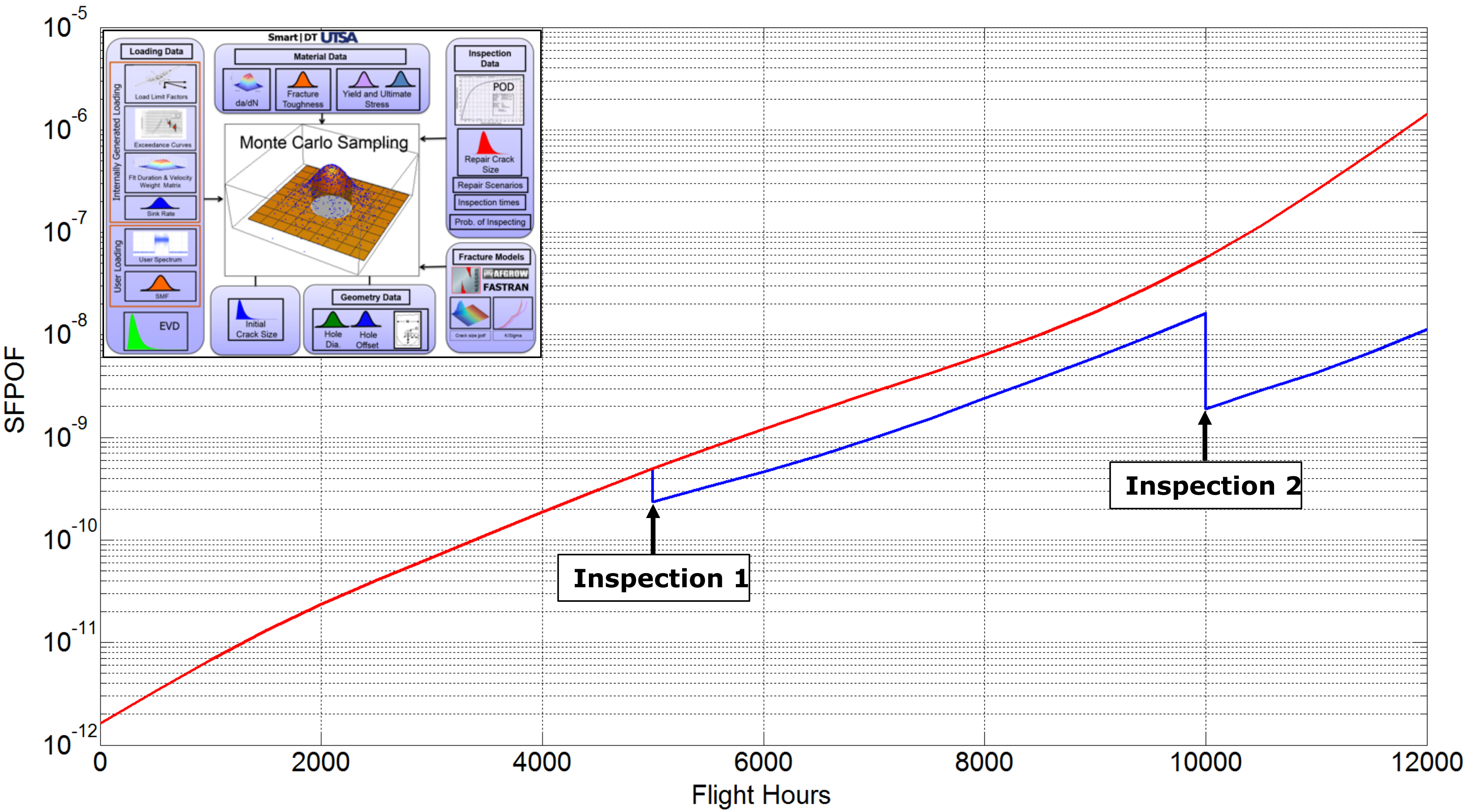
$$Hz(t) = \frac{POF_{\text{surv}}(t)}{1 - CTPOF(t)}$$

$F_{EVD}$ = CDF of maximum stress per flight (exteme value distribution).



# PDTA Results: SFPOF

## (SFPOF: Single Flight Probability Of Failure)

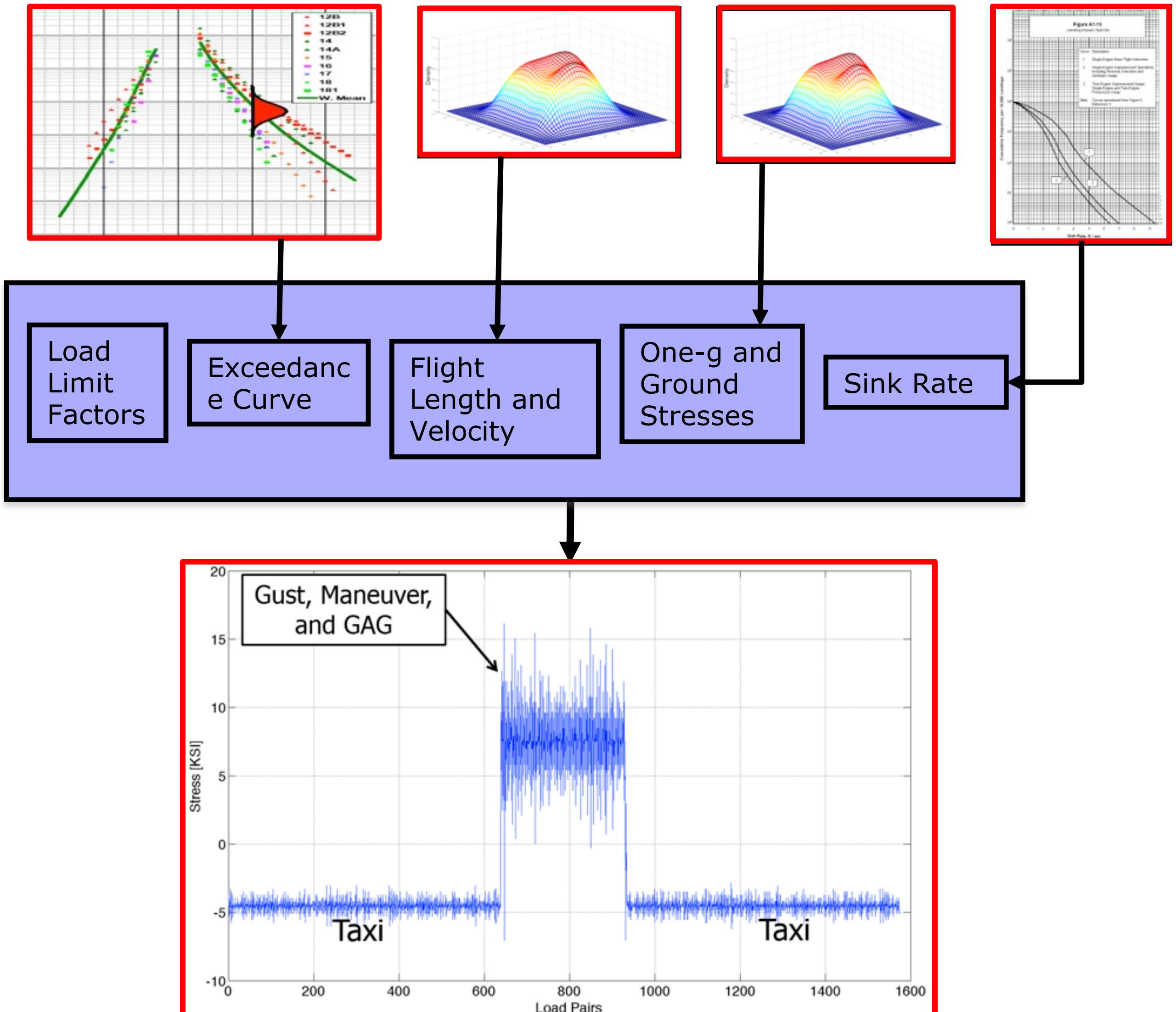


# Random Variable Summary


| Random Variable | SMART\|DT Options |
|---|---|
| Initial Crack Size | Lognormal, Weibull, Tabular, Tabular joint a and c |
| Fracture Toughness | Normal, Tabular |
| Extreme Load per Flight | Gumbel, Weibull, Frechet |
| da/dN Parameters | Correlated normal |
| Crack Aspect Ratio | Normal, Tabular |
| Hole Diameter | Normal, Tabular |
| Hole Offset | Normal |
| Yield Stress | Normal |
| Ultimate Stress | Normal |
| Peak Stress | Uniform |

Expandable

# Spectrum Generation




**SMART | DT Features**

- ❑ Exceedance curves
  - Internal and user-defined
  - Mixed usages
- ❑ Flight duration and weight matrices random to simulate flight profiles or different operations
- ❑ Randomized flights and stresses
- ❑ Spectrum editing options
- ❑ User-defined spectra
  - Afgrow format

# SMART|DT Loading Exceedance Options

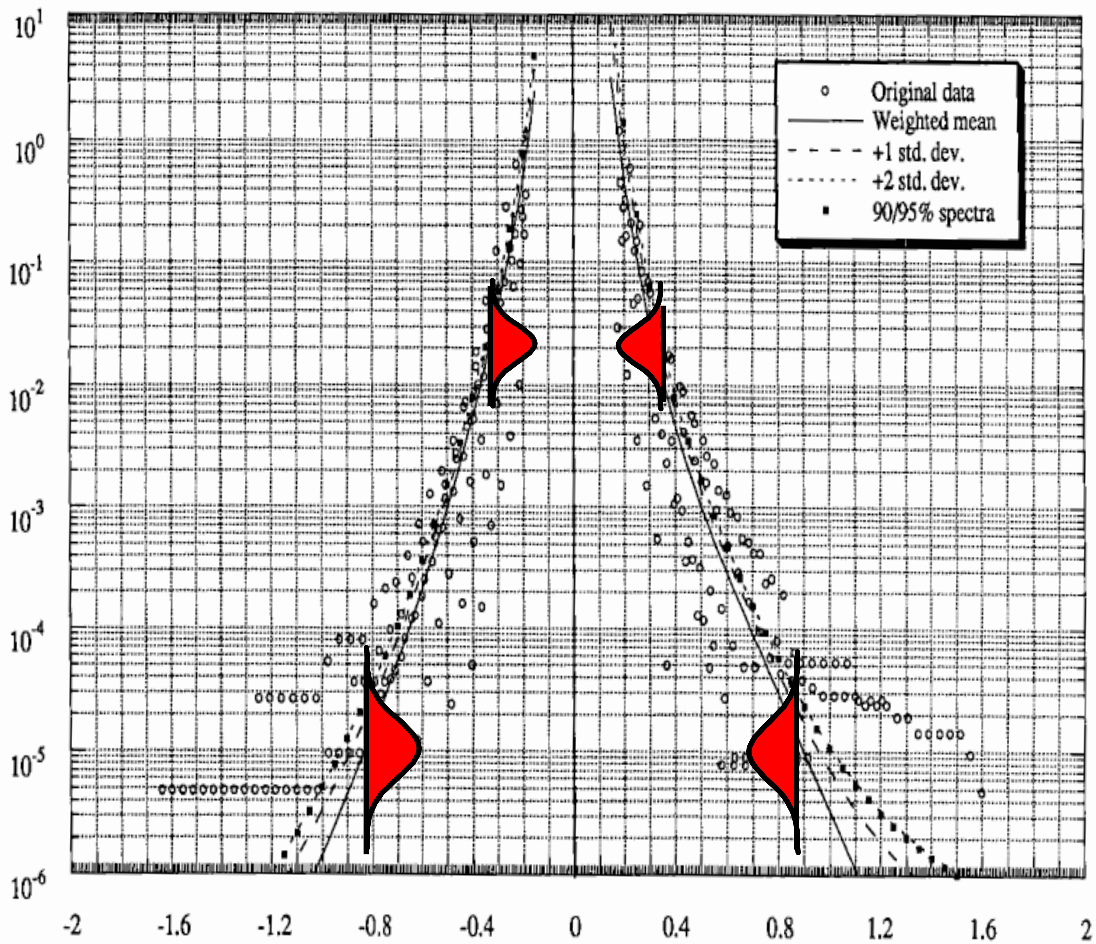

| Usages |
| --- |
| Single-Engine Unpressurized Usage Basic Flight Instruction |
| Single-Engine Unpressurized Usage Personal Usage |
| Single-Engine Unpressurized Usage Executive Usage |
| Single-Engine Unpressurized Usage Aerobatic Usage |
| Twin-Engine Unpressurized Usage Basic Flight Instruction |
| Twin-Engine Unpressurized Usage General |
| Pressurized Usage |
| Agricultural/Special Usage |
| **User defined** |

Note: exceedance data are normalized to velocity and limit load factor

Mix of weighted usages allowed

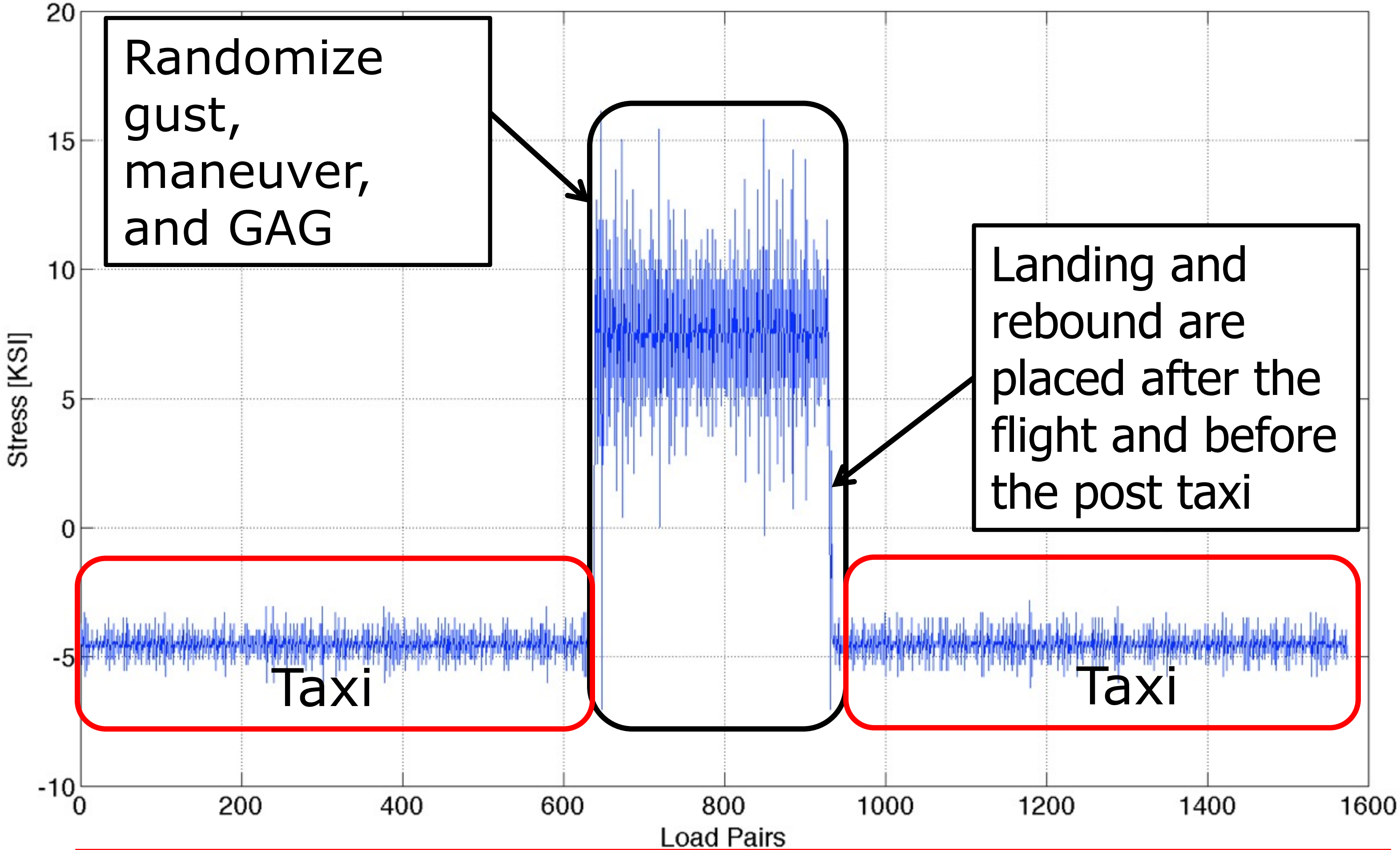
# Loading Generation

Exceedances/
Nautical Mile




Acceleration Fraction, $a_n/a_{n_{LLF}}$

# Loading Example



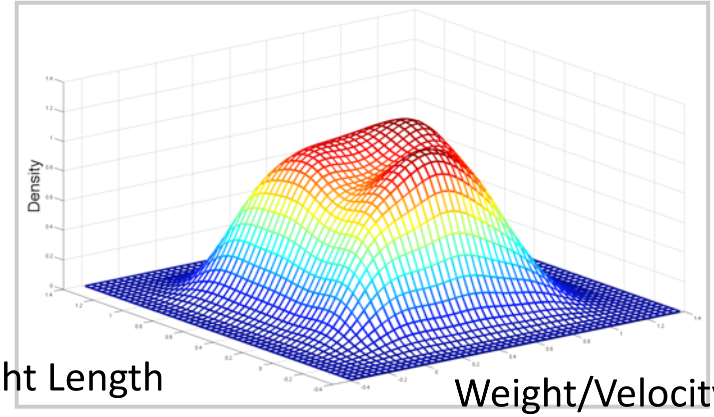

Randomize taxi loads and split half before the flight and half after the flight. Taxi load can be excluded from the analysis.

# Usage and Loads

- SMART uses two matrices to account for variation in:
  - Flight length
  - Flight velocity
  - Flight weight
- Matrices can describe fleet variation, or
- Matrices can describe flight profiles

**Flight Length and Weight/Velocity Matrix**

| Flight time (Hours) | % of Flights | Weight (1g_stress and Ground_stress) Percentage | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1.00 | 0.95 | 0.90 | 0.85 | 0.80 | 0.75 | 0.70 |
| 0.25 | 0.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.50 | 0.05 | 0 | 0 | 0.05 | 0.25 | 0.6 | 0.1 | 0 |
| 0.75 | 0.15 | 0 | 0 | 0.25 | 0.4 | 0.3 | 0.05 | 0 |
| 1.00 | 0.35 | 0.05 | 0.15 | 0.45 | 0.3 | 0.05 | 0 | 0 |
| 1.25 | 0.10 | 0.05 | 0.15 | 0.45 | 0.3 | 0.05 | 0 | 0 |
| 1.50 | 0.10 | 0.05 | 0.3 | 0.5 | 0.15 | 0 | 0 | 0 |
| 1.75 | 0.20 | 0.05 | 0.3 | 0.5 | 0.15 | 0 | 0 | 0 |
| 2.00 | 0.05 | 0.15 | 0.55 | 0.2 | 0.1 | 0 | 0 | 0 |



# Usage Variation - Fleet

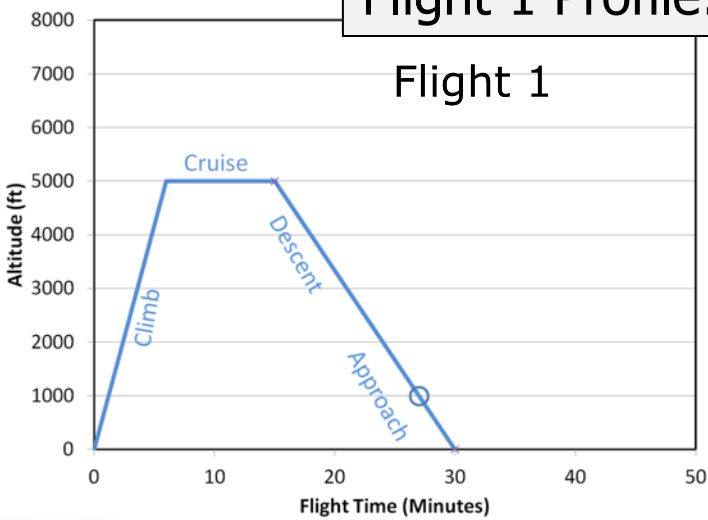
| Mission | Mission Name | % of flights | Flight Duration (hr) | MTOW (lb) | Cruise Speed (Kts) |
|---|---|---|---|---|---|
| A | Check ride | 10% | 0.2 | 5200 | 160 |
| B | High speed cruise | 20% | 0.9 | 6800 | 180 |
| C | Max weight | 30% | 1.1 | 7000 | 175 |
| D | Max range | 40% | 3 | 6600 | 170 |

| Mission | Flight time (Hours) | % of Flights (sums to 1.0) | Average Speed During Flight, % Design Velocity | | | |
|---|---|---|---|---|---|---|
| | | | 0.8 | 0.9 | 0.0875 | 0.85 |
| | | | % of Flights (sums to 1.0) | | | |
| A | 0.2 | 0.1 | 1.0 | 0 | 0 | 0 |
| B | 0.9 | 0.2 | 0 | 1.0 | 0 | 0 |
| C | 1.1 | 0.3 | 0 | 0 | 1.0 | 0 |
| D | 3.0 | 0.4 | 0 | 0 | 0 | 1.0 |

| | Flight time (Hours) | % of Flights (sums to 1.0) | Average Weight During Flight, % Design Weight | | | |
|---|---|---|---|---|---|---|
| | | | 0.74 | 0.97 | 1.00 | 0.94 |
| | | | % of Flights (sums to 1.0) | | | |
| A | 0.2 | 0.1 | 1.0 | 0 | 0 | 0 |
| B | 0.9 | 0.2 | 0 | 1.0 | 0 | 0 |
| C | 1.1 | 0.3 | 0 | 0 | 1.0 | 0 |
| D | 3.0 | 0.4 | 0 | 0 | 0 | 1.0 |

# Usage Variation –Flight Profiles

## Flight 1 Profile: TE unpress. general usage

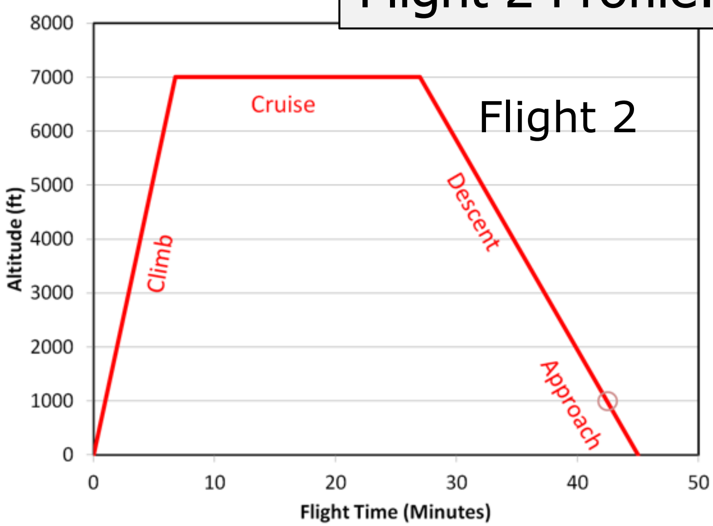



| Segment | Weight | KEAS | % Duration |
|---|---|---|---|
| CLIMB | 6580 | 140 | 0.20 |
| CRUISE | 6440 | 160 | 0.50 |
| DESCENT | 6300 | 180 | 0.20 |
| APPROACH | 6160 | 120 | 0.10 |

$V_C$ = 200 KEAS

MTOW = 7000 lb

## Flight 2 Profile: TE unpress. general usage




| Segment | Weight | KEAS | % Duration |
|---|---|---|---|
| CLIMB | 6720 | 140 | 0.15 |
| CRUISE | 6580 | 160 | 0.60 |
| DESCENT | 6440 | 180 | 0.15 |
| APPROACH | 6300 | 120 | 0.10 |

$V_C$ = 200 KEAS

MTOW = 7000 lb

13

# Usage Variation – Flight Profiles

## Flight 1 Profile: TE unpress. general usage

| Flight time (Hours) | % of Flights (sums to 1.0) | Average Speed During Flight, % Design Velocity | | | | |
|---|---|---|---|---|---|---|
| | | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| | | % of Flights (sums to 1.0) | | | | |
| 0.5 | 1 | 0.1 | 0.2 | 0.5 | 0.2 | 0 |

| Flight time (Hours) | % of Flights (sums to 1.0) | Average Weight During Flight, % Design Weight | | | | |
|---|---|---|---|---|---|---|
| | | 0.88 | 0.9 | 0.92 | 0.94 | 0.96 |
| | | % of Flights (sums to 1.0) | | | | |
| 0.5 | 1 | 0.1 | 0.2 | 0.5 | 0.2 | 0 |

## Flight 2 Profile: TE unpress. general usage

| Flight time (Hours) | % of Flights (sums to 1.0) | Average Speed During Flight, % Design Velocity | | | | |
|---|---|---|---|---|---|---|
| | | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| | | % of Flights (sums to 1.0) | | | | |
| 0.75 | 1 | 0.1 | 0.15 | 0.6 | 0.15 | 0 |

| Flight time (Hours) | % of Flights (sums to 1.0) | Average Weight During Flight, % Design Weight | | | | |
|---|---|---|---|---|---|---|
| | | 0.88 | 0.9 | 0.92 | 0.94 | 0.96 |
| | | % of Flights (sums to 1.0) | | | | |
| 0.75 | 1 | 0 | 0.1 | 0.15 | 0.6 | 0.15 |

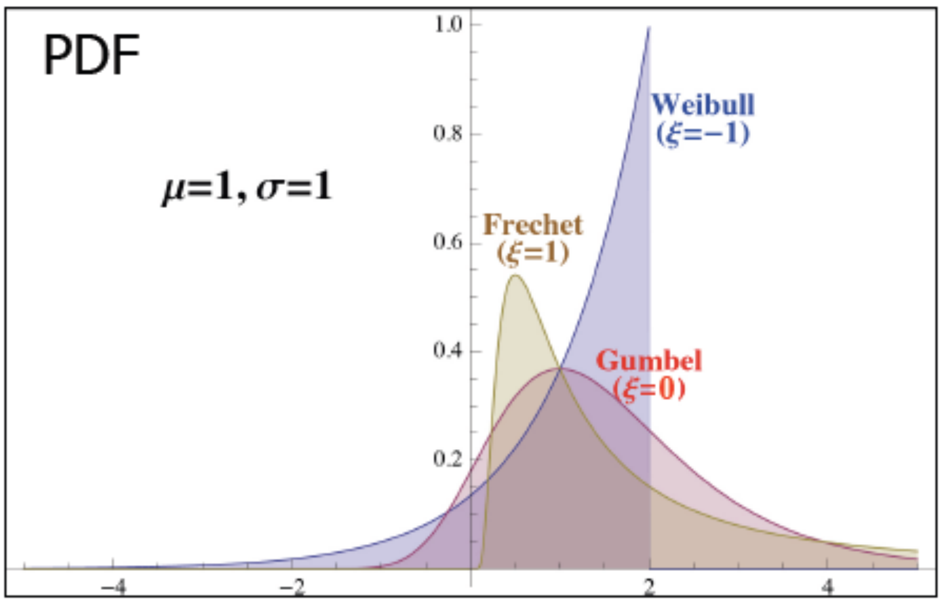
# EVD Generation

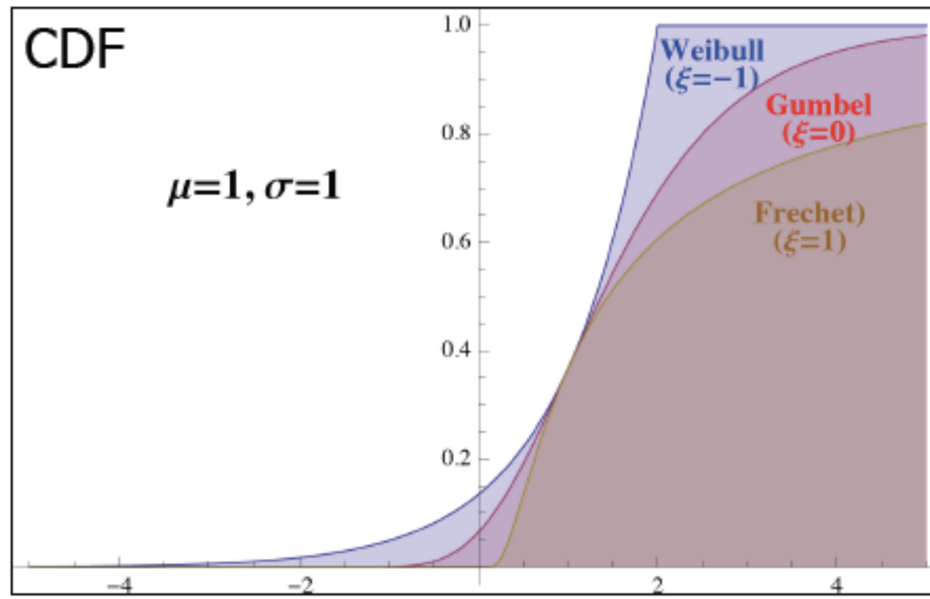
## (Extreme Value Distribution)

- Maximum Weibull, Frechet, or Gumbel can be written in terms of the Generalized Extreme Value Distribution as

$$F(x) = \exp\left\{ -\left[ 1 + \xi\left( \frac{x-\mu}{\sigma} \right) \right]^{-1/\xi} \right\} \qquad \begin{array}{ll} \xi = 0 & \text{Gumbel} \\ \xi > 0 & \text{Frechet} \\ \xi < 0 & \text{Weibull} \end{array}$$

- Parameters $(\mu, \sigma, \xi)$ location, scale, and shape define the distribution.

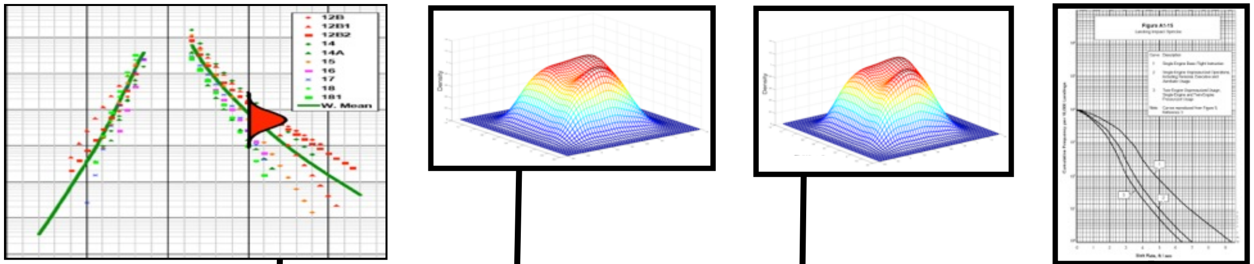



Maximum likelihood estimation

# EVD Generation

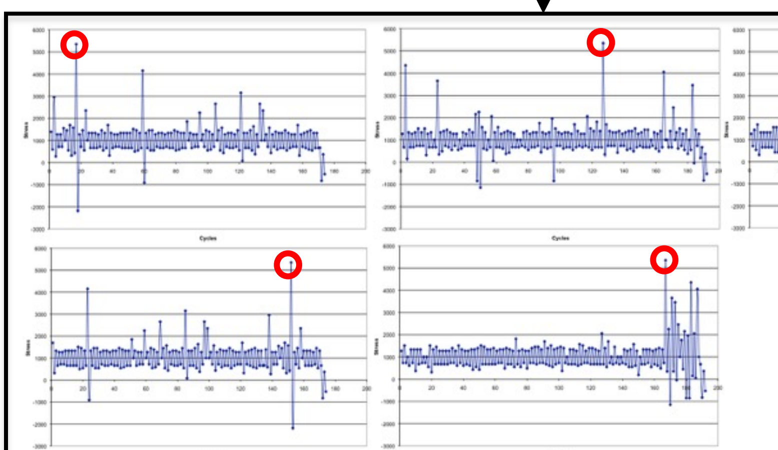
## (Extreme Value Distribution)



- Load Limit Factors
- Exceedance Curve
- Flight Length and Velocity
- One-g and Ground Stresses
- Sink Rate

✓ EVD computed internally from spectrum

$$F(x) = \exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$



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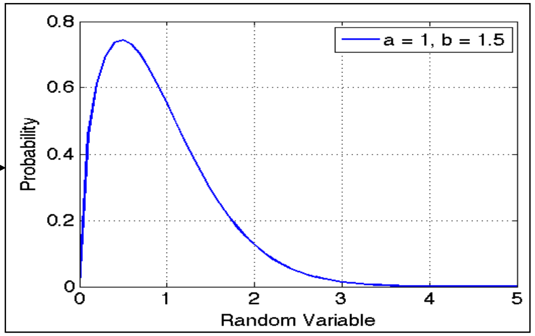
*n* flights until EVD converges

## EVD Distribution



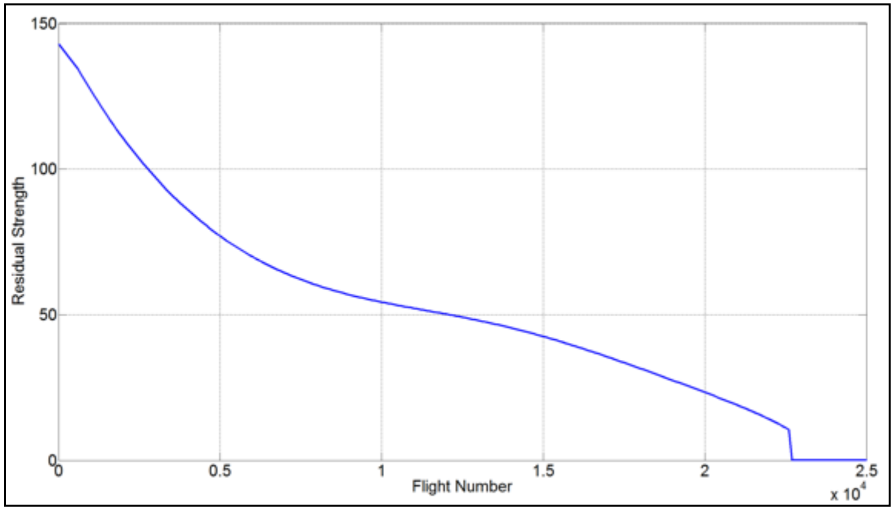
# Residual Strength Interpolation

- From Fracture Mechanics we know:

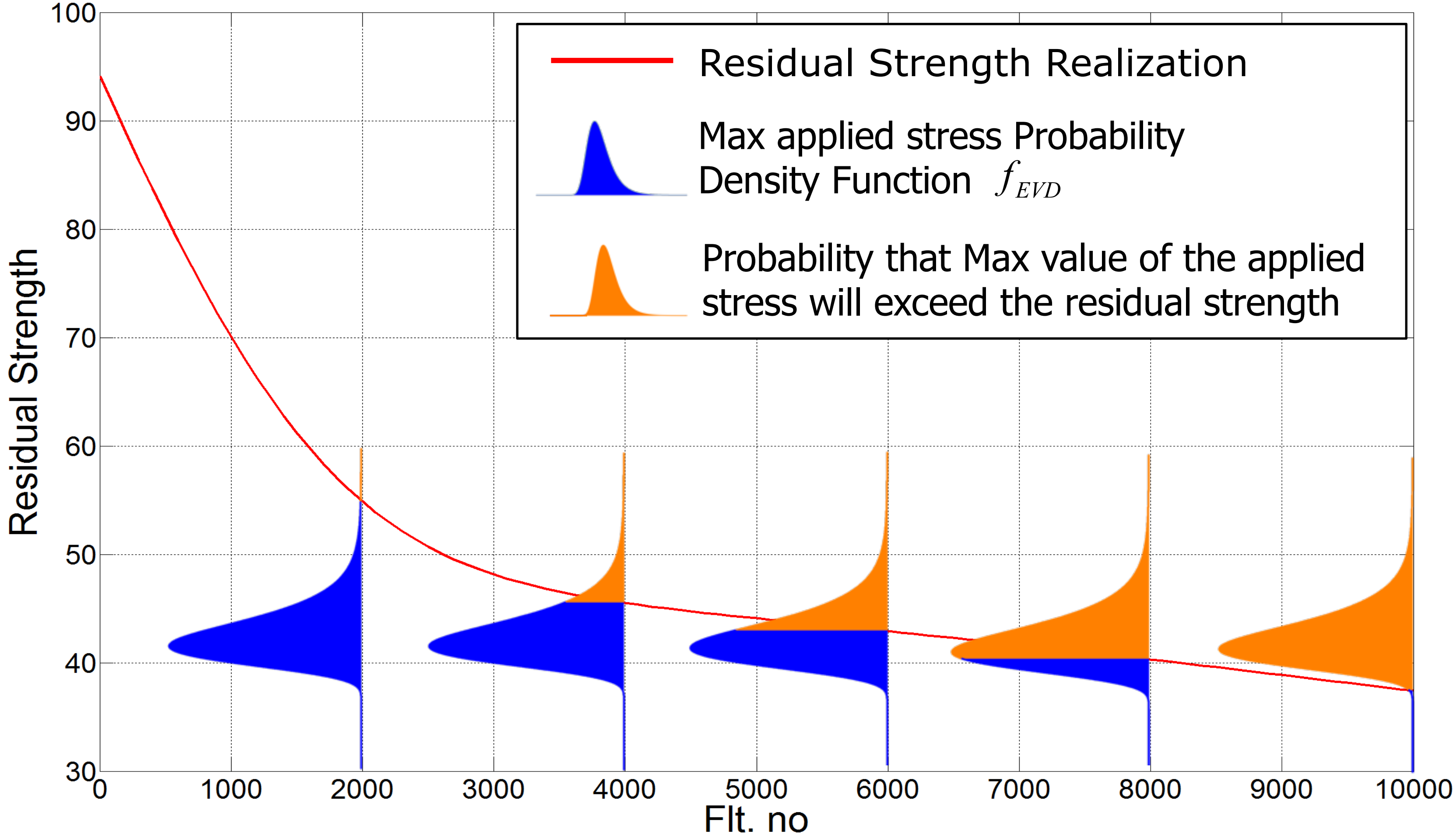
$$K_C = \sigma_{RS}\beta(a(a_o,t))\sqrt{\pi a(a_o,t)}$$

- Residual Strength can be defined as:

$$\sigma_{RS} = \frac{K_C}{\beta(a(a_o,t))\sqrt{\pi a(a_o,t)}}$$



# Probability of Failure



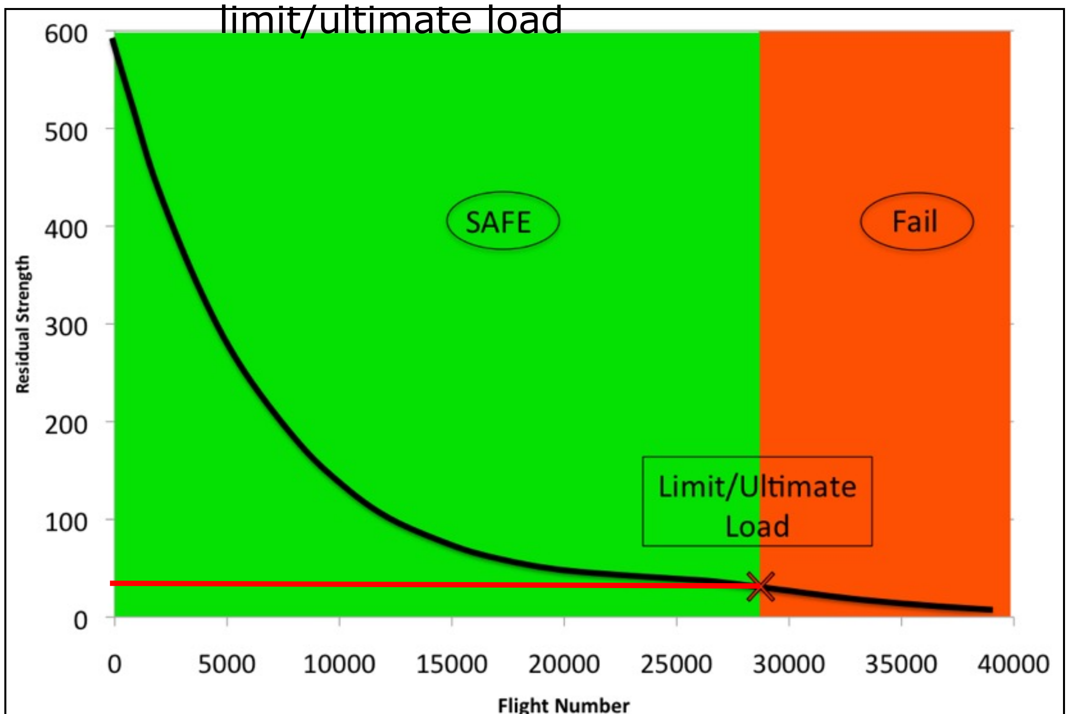

$$POF(t|a_i K_C) = 1 - F_{EVD}\left(\frac{K_C}{\beta(a(a_o,t))\sqrt{\pi a(a_o,t)}}\right)$$

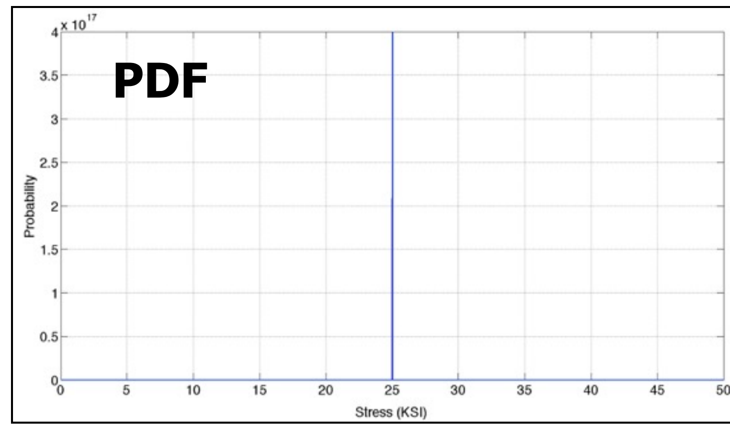
# Limit/Ultimate Load EVD

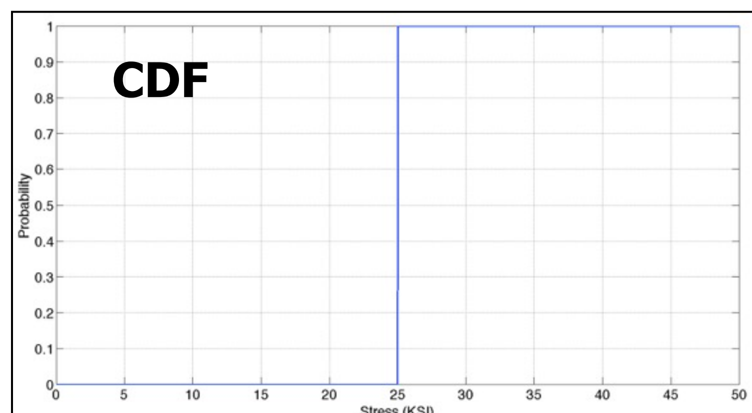
Smart|DT allows the user to input the limit load as a deterministic EVD input.

- Residual strength ≤ limit load has a POF = 1
- Residual strength > limit load has a POF = 0

EVD is set to a deterministic value equal to the airplane limit/ultimate load



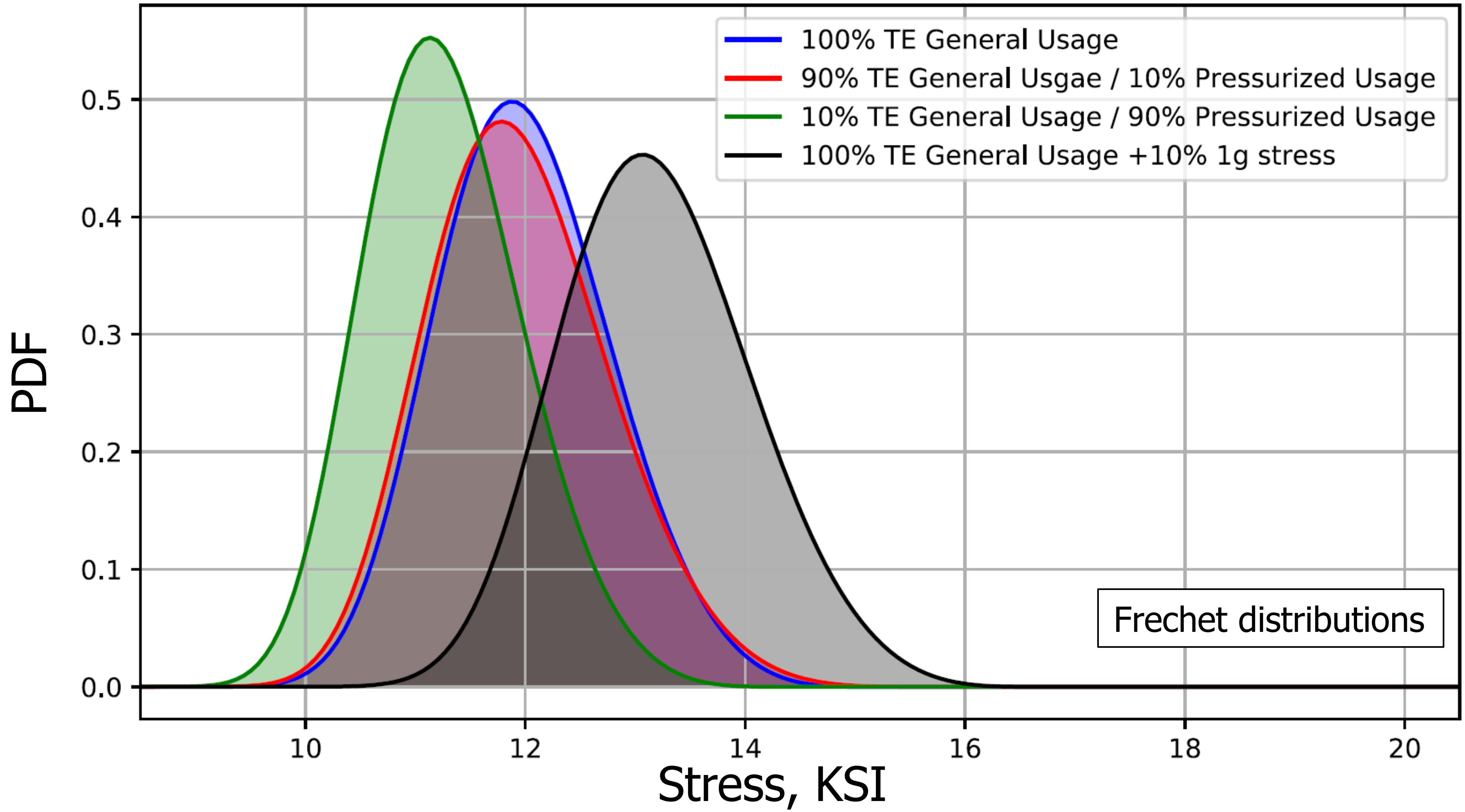



# Comparison of EVD

| Usage | EVD Parameters | | |
|---|---|---|---|
| | Location $\mu$ | Scale $\sigma$ | Shape $\xi$ |
| 100% TE General Usage | 11.697 | 0.757 | 0.218 |
| 90% TE General Usage/10% Pressurized usage | 11.625 | 0.779 | 0.187 |
| 10% TE General Usage/90% Pressurized usage | 10.984 | 0.680 | 0.197 |
| (All 1g stress = 5700 psi) | | | |
| | | | |
| 100% TE General Usage +10% 1g stress | 12.866 | 0.833 | 0.218 |

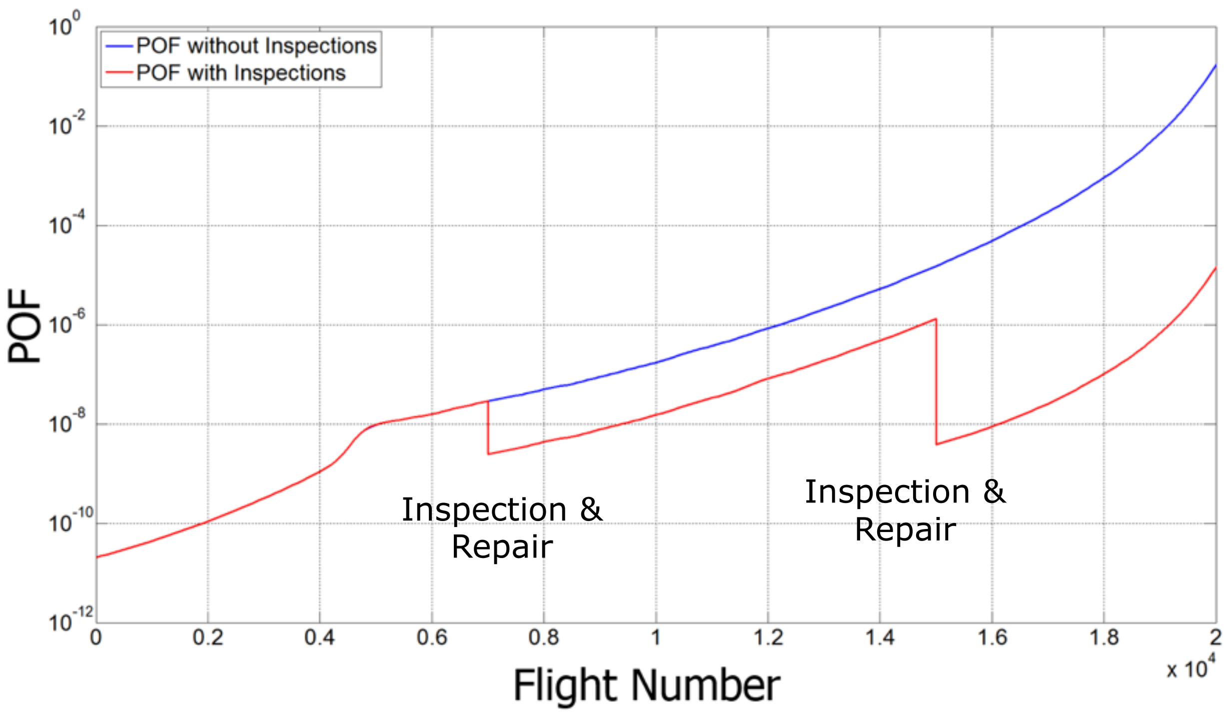
EVD: Extreme Value Distribution

# Comparison of EVD



# Inspections

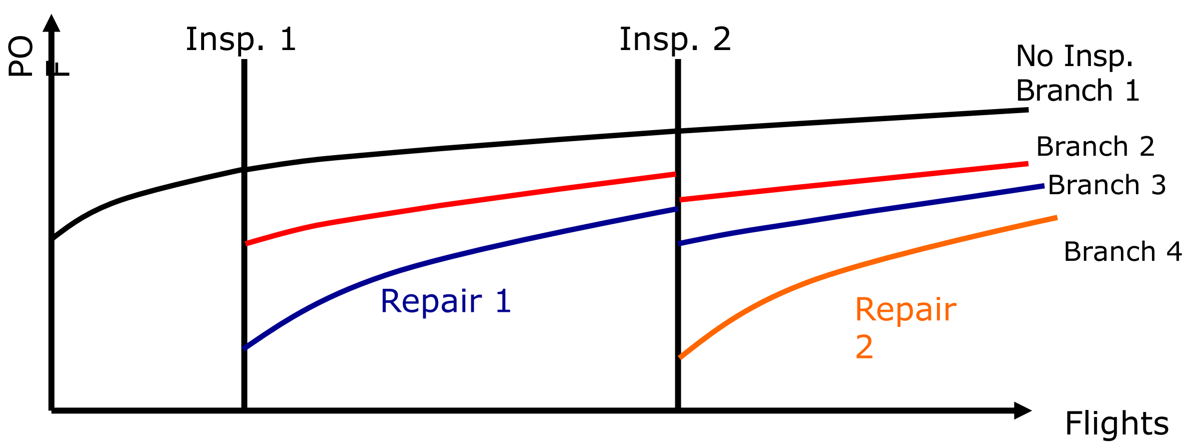
- ❑Probability of Detection (POD)
- ❑Probability of Inspection (POI)
- ❑Inspection times
- ❑After inspection and/or repair crack size



# SMART|DT Methodology

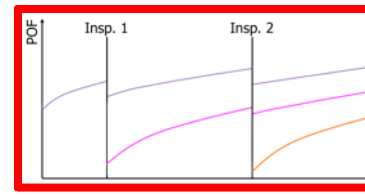
- Treat inspections as multiple "branches" where each branch represents a repair scenario.
- Each branch computed independently.
- Overall POF determined as a sum from all branches.

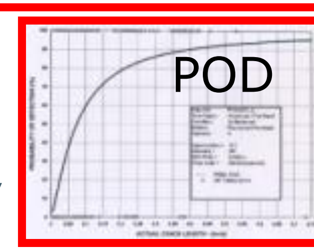
POD, POI and "after repair initial crack size" can be changed for each branch (different repair scenarios can be analyzed).

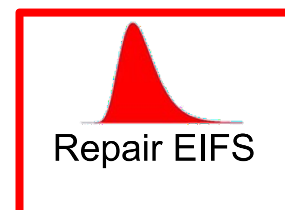


# Inspection Capabilities

- Smart|DT has robust inspection and repair capabilities:
  - Any number of inspections at user-defined flights
  - Different scenarios for each inspection
  - POD curve inputs (deterministic, tabular, lognormal)
  - Probability of Inspection
  - Arbitrary repair EIFS (deterministic, tabular, lognormal, Weibull)

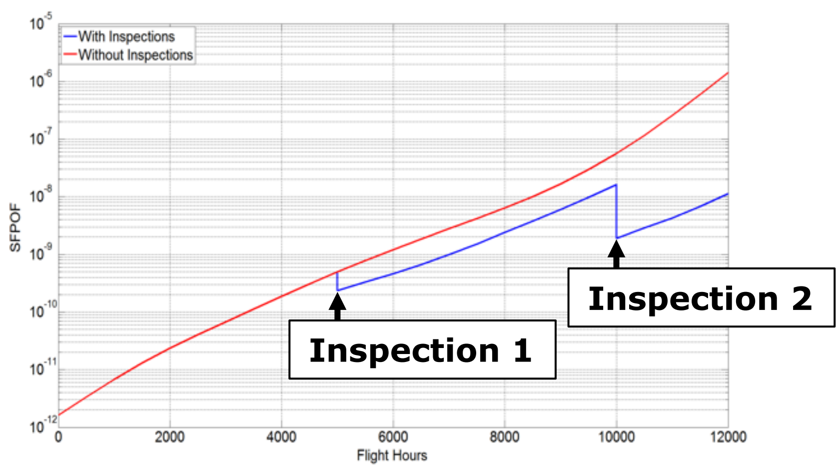


# Summary

- PDTA quantifies risk by considering variation in initial flaw size, material, geometry, usage, inspection, etc.
- PDTA tools are useful to assess usage severity variation
  - Especially important when assessing in-service issues
- PDTA tools incorporate effects of inspections and repair to assess risk for various repair scenarios



# Monte Carlo Sampling

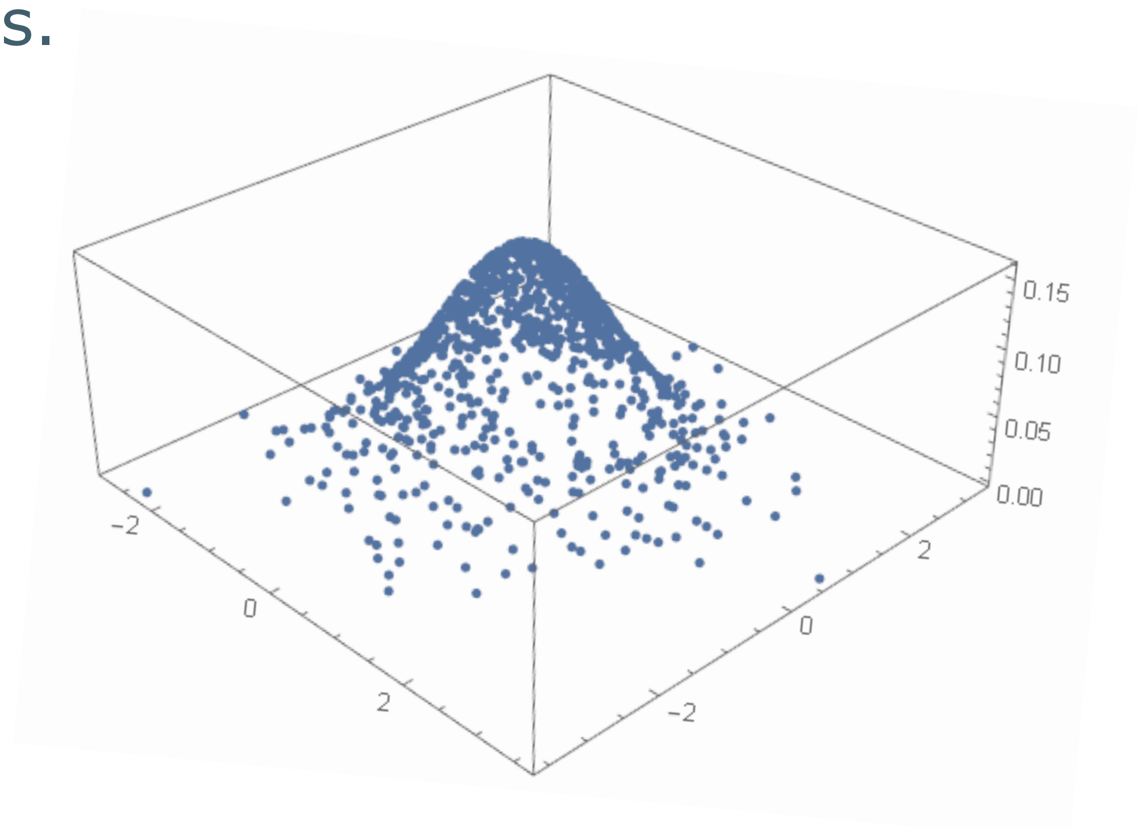
# Outline

- ✓Basics of Monte Carlo sampling
  - ✓Limit state
  - ✓Indicator function
  - ✓Academic Excel example
    - ✓How to generate samples to compute pi (3.1416)
- ✓PDTA Example (Generate samples for Kc and MaxLoad)
  - ✓Define limit state
  - ✓Setting confidence limits as a function of the number of samples


27

# Monte Carlo Sampling

Monte Carlo sampling is a technique to evaluate difficult integrals (multi-dimensional) or to sample random variables governed by probability density functions.



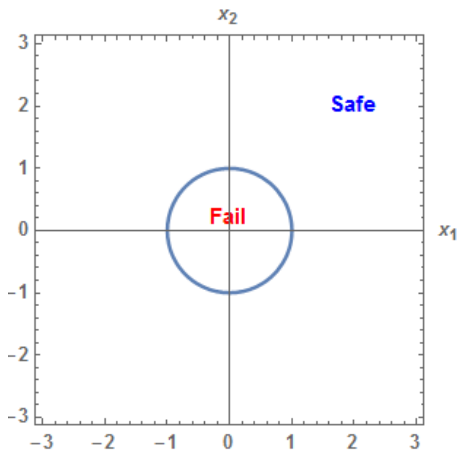
# The Limit State

The limit state “g” is used to define the failure domain. The definition of failure is always rewritten such that:

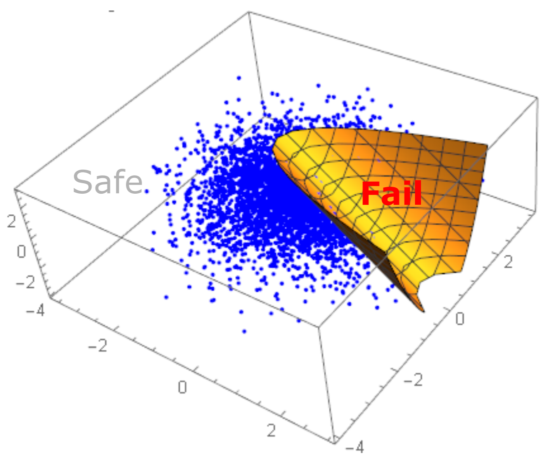
$$g(x) = 0 \quad \text{- limit state}$$

$$g(x) \leq 0 \quad \text{- failure}$$

$$g(x) > 0 \quad \text{- safe}$$




$$x_1^2 + x_2^2 - 1 = 0$$




$$\frac{x_3}{2} - x_2^2 + x_1 - 1 = 0$$

# Typical engineering examples of limit states

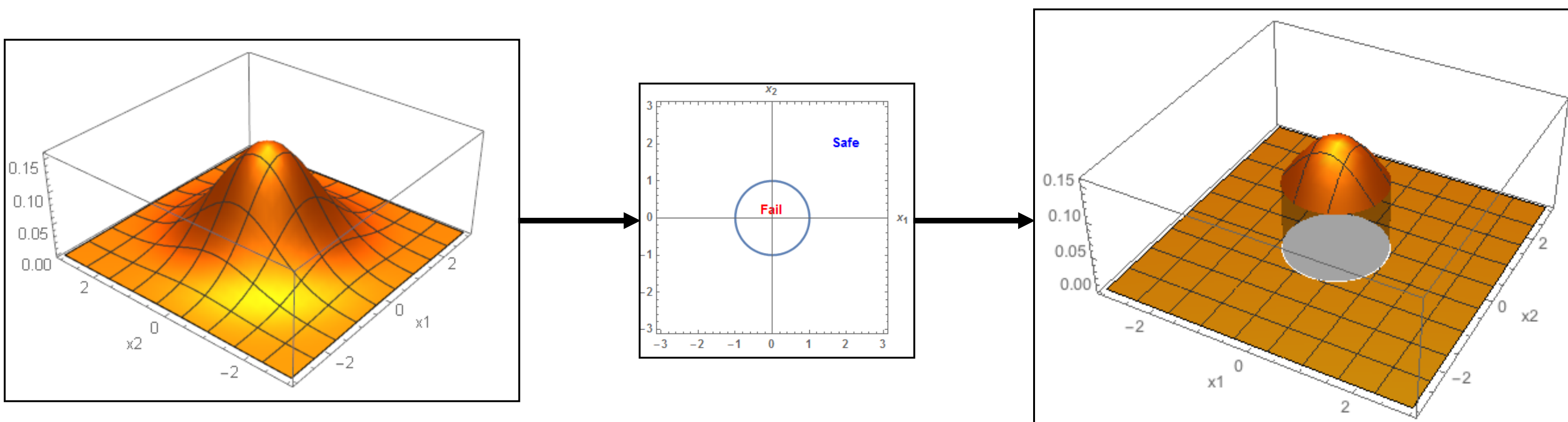
- ✓yield stress < stress
- ✓clearance < max allowable displacement
- ✓fracture toughness < stress intensity factor
- ✓critical crack size < growing crack size
- ✓material thickness < corrosion depth
- ✓vibration amplitude < max. allowable amp.

# Indicator Function

To compute the probability of failure, the binary indicator function is used:

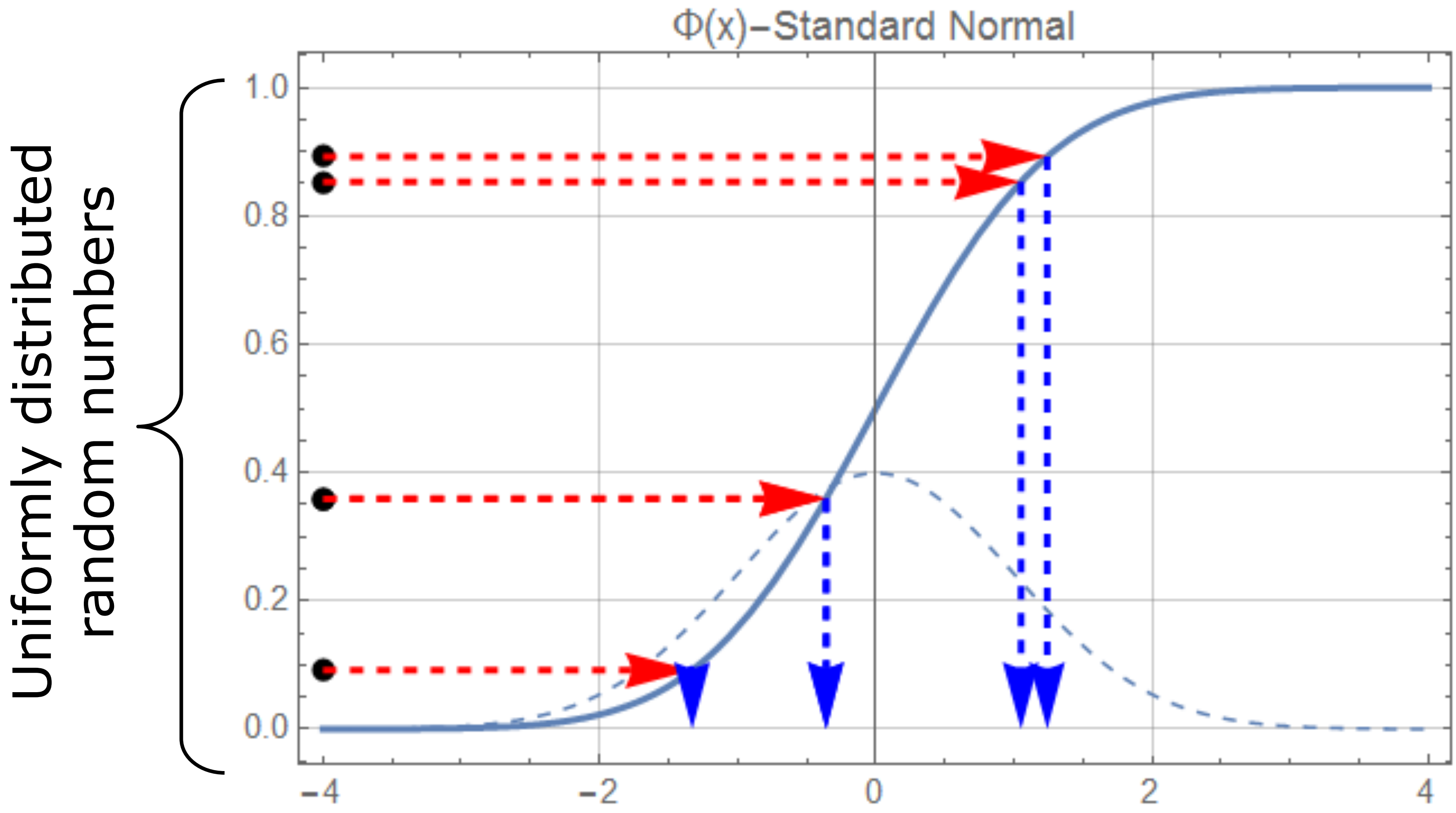
$$P_f = \int_{-\infty}^{\infty} \mathrm{I}(\mathbf{x})\, f_{\mathbf{x}}(\mathbf{x})\, d\mathbf{x}$$

The indicator function is a binary, 1 or 0, function defined as equal to "1" in the failure domain and "0" in the safe domain

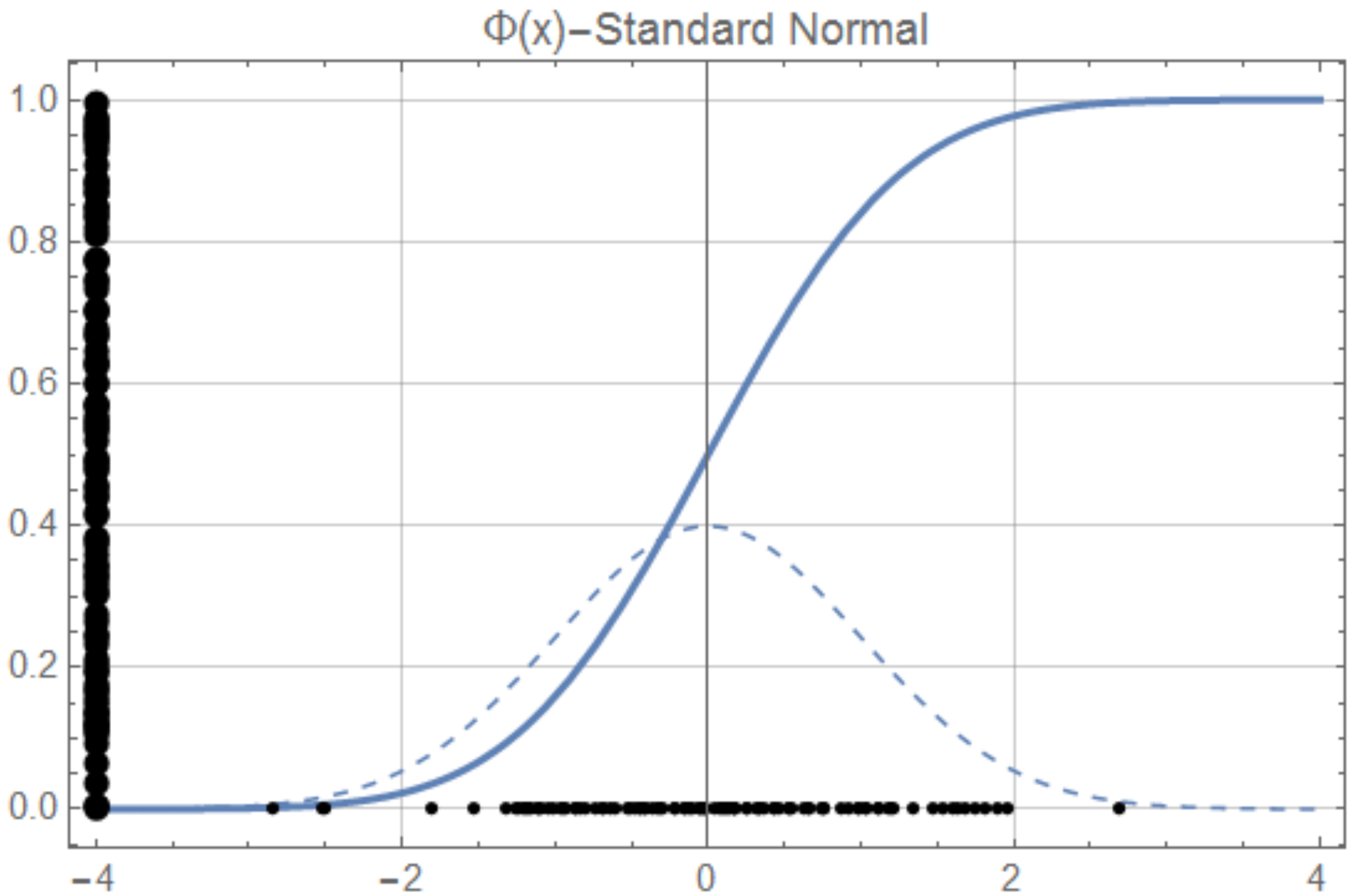


# How to Generate Samples

Random Sampling (Inverse Integral Method)

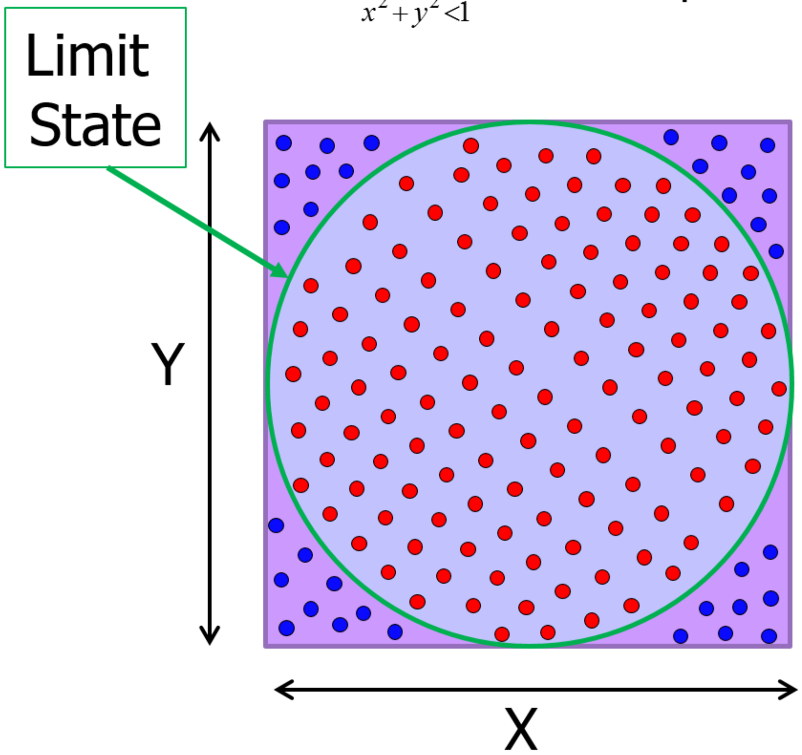


# How to Generate Samples



# A simple Monte Carlo Example

$$A = \int\int_{x^2+y^2<1} dxdy = \frac{\pi}{4}$$



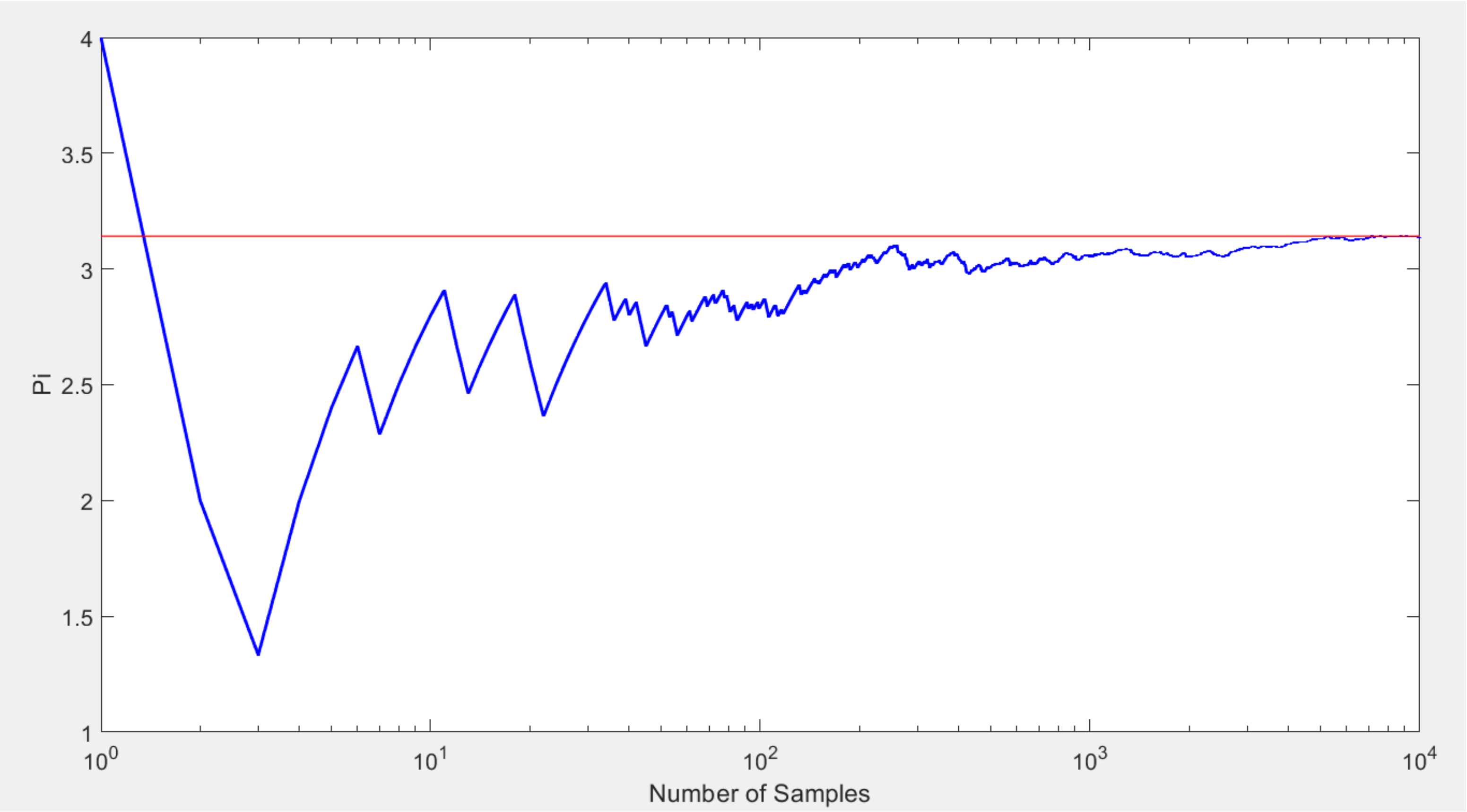

$$x_1^2 + x_2^2 - 1 \le 0$$

How Monte Carlo Works:

1. Randomly select a large number of points inside the square (Domain).
2. Count how many points lands inside the circle
3. Divide by the total number of points on the domain
4. The answer is exact when the number of points goes to infinite

Excel exercise

# Convergence wrt Number of Samples




Excel exercise

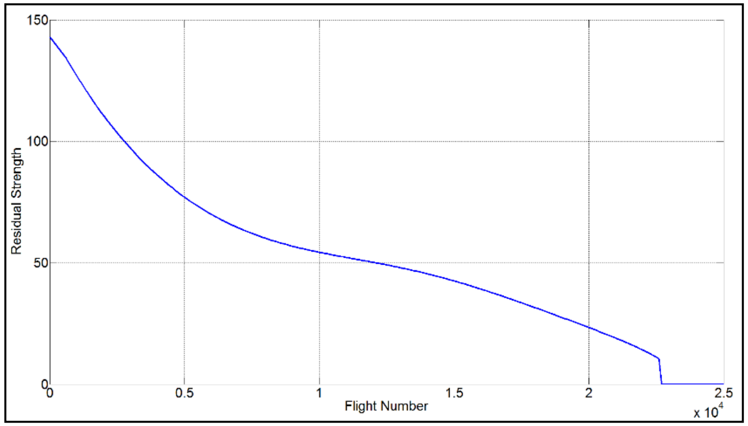
# PDTA Example

The Residual Strength (RS) at a given time is:

$$\sigma_{RS}(t) = \frac{K_C}{\beta(a(a_o,t))\sqrt{\pi a(a_o,t)}}$$

Having the maximum load per flight, the limit state is:

$$g(x) \Rightarrow \sigma_{RS}(t) - MaxLoad < 0$$



# PDTA Example


The Residual Strength (RS) at a given time is:

$$\sigma_{RS}(t) = \frac{K_C}{\beta(a(a_o,t))\sqrt{\pi a(a_o,t)}}$$

Having the maximum load per flight, the limit state is:

$$g(x) \rightarrow \sigma_{RS}(t) - MaxLoad < 0$$

Excel exercise

# PDTA Example


Steps:

1. Generate MaxLoad realizations G~(0.8,0.5)
2. Generate Fracture Toughness (Kc) realizations N~(34.5,3.8)
3. Generate Residual Strength as:

$$\sigma_{RS}(t) = \frac{K_{c_i}}{\alpha(8000\ FH)} = \frac{K_{c_i}}{2.8}$$

4. Evaluate the limit state using the indicator function

If $\sigma_{RS}(t) > MaxLoad\ \rightarrow I(x) = 0$

If $\sigma_{RS}(t) < MaxLoad\ \rightarrow I(x) = 1$

5. Count the number of failures (Sum(I(x)) and compute the POF as

$$POF(t) = \frac{sum(I(x))}{\#\ Sampoes}$$


38

# POF Convergence wrt Number of Samples

- The standard deviation of the probability estimate reduces as the square root of the number of samples.

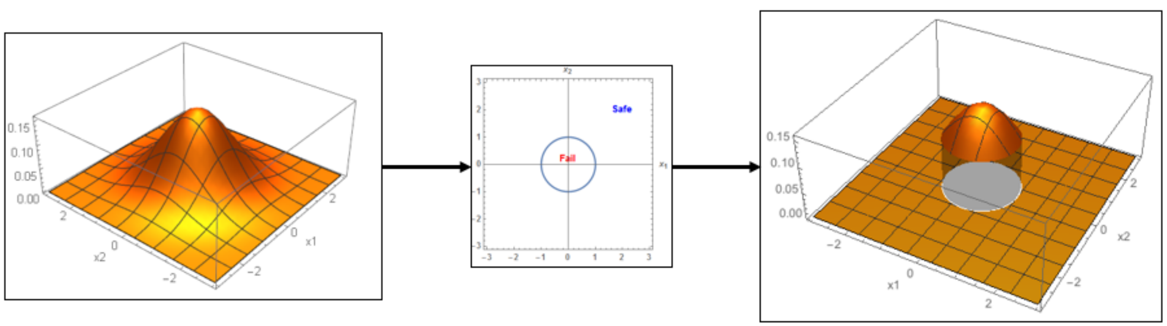
$$\sigma_{\bar{P}} = \sqrt{\frac{\bar{P}(1-\bar{P})}{N}}$$

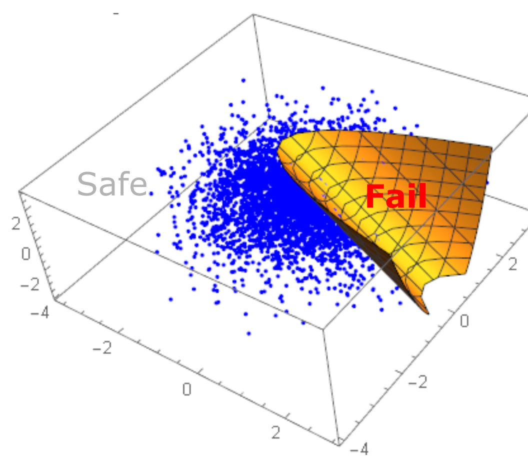
$$\delta_{\bar{P}} = \sqrt{\frac{(1-\bar{P})}{\bar{P}N}}$$

$$N = \frac{1-\bar{P}}{\bar{P}\delta_{\bar{P}}}$$

# Summary

- Basics of Monte Carlo sampling were reviewed
  - Limit state and Indicator function
- PDTA Example (Generate samples for Kc)
  - Define limit state
  - Setting confidence limits as a function of the number of samples




$$P_f = \int_{-\infty}^{\infty} \mathrm{I}(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}$$

# Questions